A General Small Cancellation Theory

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Abstract

In this article a generalized version of small cancellation theory is developed which is applicable to specific types of high-dimensional simplicial complexes. The usual results on small cancellation groups are then shown to hold in this new setting with only slight modifications. For example, arbitrary dimensional versions of the Poincaré construction and the Cayley complex are described.

0.1 Main Theorems

In this article a generalized version of traditional small cancellation theory is developed which is applicable to specific types of high-dimensional simplicial complexes. The usual results on small cancellation groups are then shown to hold in this new setting with only slight modifications. The main results derived for this general small cancellation theory are summarized below in Theorem A. The notions of general relators, Cayley categories, and general small cancellation presentations are being introduced here, and will be defined in the course of the article.

Theorem A *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $\alpha \leq \frac{1}{12}$, then the word and conjugacy problems for G are decidable, the Cayley graph is constructible, the Cayley category of the presentation is contractible, and G is the direct limit of hyperbolic groups. If in addition, the presentation satisfies the hypotheses of Lemma 14.17, then every finite subgroup of G is a subgroup of the automorphism group of some general relator in $\mathcal{R}$.*

In a forthcoming article, the general small cancellation theory developed here will be applied to the Burnside groups of sufficiently large exponent. The main results on the Burnside groups which are proved in the article are summarized below in Theorem B. The notation $\mathcal{B}(m,n)$ refers to the m-generated free Burnside group of exponent n.

Theorem B *For any $m > 1$, and any $n \geq 1260$, the Burnside group $\mathcal{B}(m,n)$ possesses a general small cancellation presentation which satisfies all of the hypotheses and conclusions of Theorem A. In addition, every finite subgroup of the group $\mathcal{B}(m,n)$ is contained in a direct product of a dihedral of order $2n$ with a* *finite number of dihedral 2-groups whose exponent divides n. As a consequence, all of the groups $\mathcal{B}(m, n)$ are infinite.*

0.2 Key Concepts

Before beginning the full development of the theory, it seems advisable to provide a brief sketch of the key concepts used in the proof. To this end, the presentation below will avoid precise definitions in favor of rough descriptions which appeal to the intuition.

General Relators: The results obtained in this article are dependent on a type of structure called a general relator, which is introduced here for the first time. In traditional small cancellation theory the cyclically reduced relators can be viewed as a finite partition of the unit circle, with each edge labeled by a generator of the group or, alternatively, as the boundary of a unit disk with these properties. The particular generalization of small cancellation theory developed in this article is based on the idea of using "relators" whose boundaries are homotopically equivalent to the unit circle, and the general relator itself is a topological cone over its boundary. These general relators, which are perhaps best viewed as topological cones over solid tubes, contain a 1-skeleton which can be significantly more complicated than in the traditional theory, but whose local structure is less important than its global topology.

Representatives: Of particular importance in this context is the notion of the winding number of a loop in the boundary of a general relator. A winding number is definable in this situation because of the homotopic equivalence to the unit circle. A loop in the boundary of a general relator with winding number 1 is called a representative of the general relator. Using representatives, it is possible to define more or less traditional van Kampen diagrams over collections of general relators by requiring that the label of every 2-cell in the planar van Kampen diagram be the label of a representative loop in the boundary of some general relator.

General Presentations, Poincaré Constructions, and Cayley Categories: The group corresponding to a set of general relators is defined by forming a variation of the Poincaré construction and setting the group of the presentation equal to the fundamental group of the construction. The universal cover of this construction contains a 1-skeleton which is the Cayley graph of this group. As in the traditional theory, the universal cover of the Poincaré construction will contain multiple copies of a general relator attached to the Cayley graph by functions which agree on the 1-skeletons in their boundaries. The number of such multiplicities are governed by the size of the automorphism group of the particular general relator involved. If these multiple copies are eliminated through a suitable identification, then the resulting structure is called the Cayley category of the presentation.

General Small Cancellation Axioms: By placing sufficient restrictions on the general relators involved it is possible to mimic the traditional proof schemes of small cancellation theory. As an example, the general relators used are required to be thin in the sense that, given any point in a relator and an arbitrary representative of the relator, the distance from the point to the representative is small compared with the minimum length of a representative. Because of this fact, it is possible to speak in a loose way of the distance traveled around the boundary of a general relator. In terms of this distance-like function it is assumed that if one general relator contains more than a specified fraction of the boundary of the other general relator then the boundary of the first relator actually contains the entire second relator.

0.3 Overview

The article is divided into five parts. The first part develops a theory of structures iteratively built out of cones. These structures are a conical version of CW complexes. The second part considers only those conical structures which topologically resemble the 2-cells traditionally used to create Poincaré constructions from group presentations. These are called general relators. The third part of the article investigates constructions which are built out of general relators. The topics include extended versions of presentations, Poincaré constructions, and covering spaces. In Part IV, the focus is narrowed once again to consider the effect that suitably generalized small cancellation conditions have on the groups presented using general relators. A list of axioms for a general small cancellation theory is presented. Finally, in Part V, the axioms are used to prove the remaining results listed in Theorem A.

0.4 Acknowledgements

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