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The BNS-invariant for the pure braid groups

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Abstract. In 1987 Bieri, Neumann and Strebel introduced a geometric invariant for discrete groups. In this article we compute and explicitly describe the BNS-invariant for the pure braid groups.

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In 1987 Robert Bieri, Walter Neumann, and Ralph Strebel introduced a geometric invariant of a discrete group that is now known as its BNS invariant [4]. For finitely generated groups the invariant is a subset of a sphere associated to the group called its *character sphere*. They proved that their invariant is an open subset of the character sphere and that it determines which subgroups containing the commutator subgroup are finitely generated. In particular, the commutator subgroup is itself finitely generated if and only if the BNS invariant is the entire character sphere. For fundamental groups of smooth compact manifolds, the BNS-invariant contains information about the existence of circle fibrations of the manifold and for fundamental groups of 3-dimensional manifolds, the BNS-invariant can be described in terms of the Thurston norm. Given these connections, it is perhaps not surprising that the BNS-invariant is typically somewhat difficult to compute. It has been completely described for some infinite families of groups, including: one-relator groups [6], right-angled Artin groups [11], and the pure symmetric automorphism groups of free groups [12]. In this article we combine aspects of the proofs of these earlier results to compute and explicitly describe the BNS-invariant for the pure braid groups.

Theorem A. *The BNS-invariant for the pure braid group P_n is the complement of a union of the P_3-circles and the P_4-circles in its character sphere. There are exactly $\binom{n}{3} + \binom{n}{4}$ such circles.*

The names "P_3-circle" and "P_4-circle" are introduced here in order to make our main result easier to state. Their definitions are given in Section 4.

Our computation of $\Sigma^1(P_n)$ has a striking connection to the previously computed resonance variety for the pure braid groups (see Proposition 6.9 in [8]). The *resonance variety* of a group is computed from the structure of its cohomology ring. (For background information and full definitions, see [15].) In fact, there are many resonance varieties just as there are many Σ invariants, but we are concerned here with the simplest forms of each. In general there is only a weak connection between the resonance variety of a group and its BNS-invariant, but when certain conditions are met, the resonance variety is contained in the complement of the BNS-invariant [14]. In some interesting cases it is known that the complement of the first resonance variety is equal to the first BNS invariant (see [13] and [7]). In section 9.9 of [15] it was asked if this equality holds for fundamental groups of complements of hyperplane arrangements in $\mathbb{C}^n$. The pure braid groups are perhaps the best known example of an arrangement group, and our result shows that this equality does hold in this case. An example presented in [16]—constructed by deleting one hyperplane from a reflection arrangement—demonstrates that this equality does not always hold for arrangement groups.

The article is structured as follows. The first three sections contain basic results about BNS-invariants, pure braid groups, and graphs. The fourth section finds several circles of characters in the complement of the invariant for the pure braid groups. The fifth section establishes a series of reduction lemmas which collectively show that every other character is contained in the invariant, thereby completing the proof.

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1. BNS invariants

In this section we recall the definition of the BNS-invariant and discuss two standard techniques used to compute them.

Definition 1.1 (BNS-invariant). Let G be a finitely generated group. A *character of G* is a group homomorphism from G to the additive reals and the set of all characters of G is an n-dimensional real vector space where n is the $\mathbb{Z}$-rank of the abelianization of G. Let $I \subset G$ be a generating set and let $\textsc{Cay}(G, I)$ denote the right Cayley graph of G with respect to I. For any character χ we let $\textsc{Cay}_\chi(G, I)$ denote the full subgraph of $\textsc{Cay}(G, I)$ determined by the vertices whose χ-values are non-negative. The property that the BNS-invariant captures is whether or not $\textsc{Cay}_\chi(G, I)$ is connected. It is somewhat surprising, but nonetheless true, that whether or not $\textsc{Cay}_\chi(G, I)$ is connected is independent of the choice of finite generating set I and thus only depends on χ. It is much easier to see that this property is preserved when χ is composed with a dilation of $\mathbb{R}$. As a consequence, one can replace characters with equivalence classes of characters where equivalence is defined by composition with dilations by positive real numbers r. The set of equivalence classes is identified with the unit sphere in $\mathbb{R}^n$ and called the *character sphere of G*:

$$S(G) = \{\chi \mid \chi \in \mathrm{Hom}(G, \mathbb{R}) - \{0\}\}/\chi \sim r \cdot \chi$$

where $r \in (0, \infty) \subset \mathbb{R}$. The *Bieri-Neumann-Strebel-invariant of G* is the set of equivalence classes of characters $[\chi]$ such that $\textsc{Cay}_\chi(G, I)$ is connected. We write $\Sigma^1(G)$ for this invariant and we write $\Sigma^1(G)^c$ for the complementary portion of the character sphere.

Remark 1.2. The superscript "1" in the notation $\Sigma^1(G)$ indicates that there are generalizations of these definitions. The first of these was introduced by Bieri and Renz in [5]. A computation of the Σ^m invariants for the pure braid groups would be very interesting, but we must say that the approach of this paper in computing $\Sigma^1(P_n)$ does not extend to these higher invariants.

These invariants have also been described in terms of Novikov homology [3], and so our result relates to the work in [9]. Bieri and Geoghegan have presented extensions of the original definition that are applicable to group actions on non-positively curved spaces [2].

We use a common algebra metaphor to describe the images of elements under χ. We say that g *lives* or *survives* if $\chi(g)$ is not zero, that g *dies* or *is killed* when $\chi(g)$ is zero, and we say that a set survives if *every* element in set survives. There are two main techniques that we use to compute BNS-invariants. One is used to show that characters are in the complement and the other is used to show that characters are in the invariant. The first is Proposition 3.3 from [4].

Lemma 1.3 (epimorphisms). *Let $\phi\colon G \twoheadrightarrow H$ be an epimorphism between finitely generated groups. If ψ is a character of H and χ is the character of G defined by $\chi = \psi \circ \phi$, then $[\chi] \in \Sigma^1(G)$ implies $[\psi] \in \Sigma^1(H)$ and $[\psi] \in \Sigma^1(H)^c$ implies $[\chi] \in \Sigma^1(G)^c$.*

Proof. If we choose generating sets I and J for G and H respectively so that $\phi(I) = J$, then the epimorphism ϕ naturally extends to a continuous map from the Cayley graph of G onto the Cayley graph of H which then restricts to a continuous map from $\textsc{Cay}_\chi(G, I)$ onto $\textsc{Cay}_\psi(H, J)$. Since the continuous image of a connected space is connected, $\textsc{Cay}_\chi(G, I)$ connected implies $\textsc{Cay}_\psi(H, J)$ is connected and $\textsc{Cay}_\psi(H, J)$ disconnected implies $\textsc{Cay}_\chi(G, I)$ is disconnected. $\square$

Lemma 1.3 is primarily used is to find characters in $\Sigma^1(G)^c$. For each homomorphism ϕ from G onto a simpler group H whose BNS-invariant is already known, the preimage of $\Sigma^1(H)^c$ under ϕ is a subset of $\Sigma^1(G)^c$. A second use of Lemma 1.3 is that it implies $\Sigma^1(G)$ is invariant under automorphisms of G. For any finitely generated group G, precomposition defines a natural right action of $\textsc{Aut}(G)$ on the character sphere with $[\chi]\cdot\alpha$ defined to be $[\chi\circ\alpha]$ for all $\alpha \in \textsc{Aut}(G)$ and all characters χ. For each automorphism $\alpha \in \textsc{Aut}(G)$, Lemma 1.3 can be applied twice, once with $\phi = \alpha$ and a second time with $\phi = \alpha^{-1}$ to obtain the following immediate corollary.

Corollary 1.4 (Automorphisms). *For any finitely generated group G, the subsets $\Sigma^1(G)$ and $\Sigma^1(G)^c$ are invariant under the natural right action of* $\textsc{Aut}(G)$ *on the character sphere of G.*

There is an alternative description of $\Sigma^1(G)$ using G-actions on $\mathbb{R}$-trees.

Definition 1.5 (Actions on $\mathbb{R}$-trees). Suppose G acts by isometries on an $\mathbb{R}$-tree T and let $\ell\colon G \to \mathbb{R}^+$ be the corresponding length function. The action is called *non-trivial* if there are no global fixed points. It is *exceptional* if there are no invariant lines. It is *abelian* if there exists a character χ of G such that the translation length function $\ell(g)$ equals the absolute value of $\chi(g)$ for all $g \in G$. When this occurs we say that this action is *associated to* χ.

The following lemma, Theorem 5.2 in [6], describes $\Sigma^1(G)$ in these terms.

Lemma 1.6 (actions and characters). *Let χ be a character of a group G. There exists an exceptional non-trivial abelian G-action on an $\mathbb{R}$-tree associated to χ if and only if $[\chi] \in \Sigma^1(G)^c$.*

For each $g \in G$, let T_g denote the *characteristic subtree* of g: when g is elliptic, T_g is its fixed point set, and when g is hyperbolic, T_g is the axis of g. There are two main facts about characteristic subtrees that we need: (1) if g and h are commuting hyperbolic isometries then $T_g = T_h$ and (2) if g commutes with a hyperbolic isometry h then $T_g \supset T_h$. Both properties are discussed in [12].

Definition 1.7 (commutation). For any subset J of a group G there is a natural graph that records which elements commute. It has a vertex set indexed by J and two distinct vertices are connected by an edge if only if the corresponding elements of $J \subset G$ commute. We call this the *commuting graph of J in G* and denote it by $C(J)$.

Definition 1.8 (domination). Let I and J be subsets of a group G. We say that J *dominates* I if every element of I commutes with some element of J. Since elements commute with themselves, this is equivalent to the assertion that every element in $I \setminus J$ commutes with some element of J.

Lemma 1.9 (connected and dominating). *Let χ be a character of a group G. If there exist subsets I and J in G such that all of J survives under χ, $C(J)$ is connected, J dominates I, and I generates G, then $[\chi] \in \Sigma^1(G)$.*

Proof. Suppose there is an abelian action of G on an $\mathbb{R}$-tree T associated to χ. Since elements of J survive under χ, each is realized as a hyperbolic isometry of the tree. Because $C(J)$ is connected, all of these isometries share a common characteristic subtree $T' = T_j$ for all $j \in J$. Because J dominates I, each element $i \in I$ commutes with a hyperbolic isometry $j \in J$ which implies $T_i \supset T_j = T'$ for all $i \in I$. Finally, I generates G, so the line T' is invariant under all of G, the action is not exceptional, and $[\chi] \in \Sigma^1(G)$ by Lemma 1.6. $\square$

Lemma 1.9 is our primary tool for finding characters in $\Sigma^1(G)$. To illustrate its utility we include an application: characters in the complement of the BNS-invariant must kill the center of the group. (We note that this application is not new; it occurs, for example, as Corollary 3.2 in the manuscript "Geometric invariants for discrete groups" by Bieri and Strebel.)

Corollary 1.10 (central elements). *If χ is a character of a group G and χ is not identically zero on the center of G then $[\chi]$ is in $\Sigma^1(G)$.*

Proof. Let I be any generating set for G and let $J = \{g\}$ where g is a central element that lives under χ. The graph $C(J)$ is connected because it only has one vertex and J dominates I because g is central. Lemma 1.9 completes the proof. □

2. Pure braid groups

Next we recall some basic properties of the pure braid groups.

Definition 2.1 (pure braid groups). Let $\mathbb{C}^n$ be an n-dimensional complex vector space with a fixed basis and let H_{ij} be the hyperplane in $\mathbb{C}^n$ defined by the equation $z_i = z_j$. The set $\{H_{ij}\}$ of all such hyperplanes is called the *braid arrangement* and it is one of the standard examples in the theory of hyperplane arrangements. The fundamental group of the complement of the union of these hyperplanes is called the *pure braid group P_n*:

$$P_n = \pi_1(\mathbb{C}^n \setminus \{H_{ij}\}).$$

Definition 2.2 (points in the plane). There is a standard 2-dimensional way to view points in the complement of the braid arrangement. For each vector in $\mathbb{C}^n$ we have a configuration of n labeled points in the complex plane. More concretely, the point p_i in $\mathbb{C}$ is meant to indicate the value of the i-th coordinate of the vector and avoiding the hyperplanes H_{ij} corresponds to configurations where these points are distinct. Paths in the hyperplane complement correspond to motions of these n labeled points in the plane which remain distinct throughout. If we trace out these motions over time in a product of $\mathbb{C}$ with a time interval, then the points become strands that braid.

Definition 2.3 (basepoint). Computing the fundamental group of a hyperplane complement requires a choice of basepoint. We select one corresponding to the configuration where the n labeled points are equally spaced around the unit circle and p_1 through p_n occur consecutively as one proceeds in a clockwise direction. See Figure 1 for an illustration. Loops representing elements of the fundamental group are motions of these points which start and end at this particular configuration.

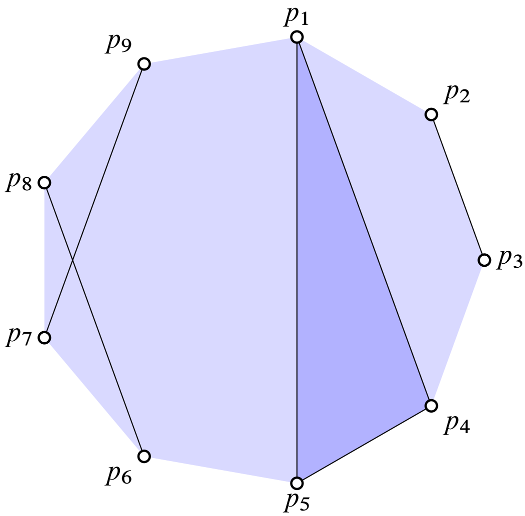

Figure 1. Nine points in convex position.

Definition 2.4 (swing generators). For each set $A \subset \{1, 2, \ldots, n\}$ of size at least 2 there is an element of P_n obtained as follows. Move the points corresponding to the elements of A directly towards the center of their convex hull. Once they are near to each other, rotate the small disk containing them one full twist in a clockwise direction and then return these points to their original position traveling back the way they came. When A is small we write S_{ij} or S_{ijk} with the subscripts indicating the points involved. In [10] these elements are called *swing generators*. One key property of the swing generator S_A is that it can be rewritten as a product of the swing generators S_{ij} with $\{i, j\} \subset A$. The order in which they are multiplied is important, but it rarely arises in this context. As an illustration of this type of factorization, the element S_{123} is equal to the product $S_{12}S_{13}S_{23}$ and to $S_{13}S_{23}S_{12}$ and to $S_{23}S_{12}S_{13}$. (For the record we are composing these elements left-to-right as is standard in the study of braid groups.) This means that the $\binom{n}{2}$ swing generators which only involve two points are sufficient to generate P_n and we call this set the *standard generating set for this arrangement*.

A presentation for the pure braid group was given by Artin in [1] and more recent geometric variations are given by Margalit and McCammond in [10]. For our purposes the most relevant fact about these various presentations is that all of their relations become trivial when abelianized. This immediately implies the following:

Lemma 2.5 (pure braid characters). *The abelianization of the pure braid group is free abelian with the images of the standard generators as a basis. As a consequence, there are no restrictions on the tuples of values a character may assign to the standard generators. Thus Hom*$(P_n, \mathbb{R}) \simeq \mathbb{R}^{\binom{n}{2}}$ *and the character sphere has dimension* $\binom{n}{2} - 1$:

$$S(P_n) = \mathbb{S}^{\binom{n}{2}-1}.$$

There are two aspects of the pure braid group that are particularly useful in this context. The first is that many pairs of swing generators commute and the second is that there is an automorphism of P_n whose net effect is to permute the labeled points in the plane without changing the character values on the corresponding standard generators.

Remark 2.6 (commuting swings). Let S_A and S_B be two swing generators in P_n. The elements S_A and S_B commute when $A \subset B$, $B \subset A$, or the convex hull of the points in A does not intersect the convex hull of the points in B [10]. For example S_{23} and S_{145} commute as do S_{14} and S_{145}, but S_{68} and S_{79} do not. See Figure 1. One consequence of this property is that the element $\Delta = S_A$ with $A = \{1, 2, \ldots, n\}$ is central in P_n. In fact Δ generates the center.

There is an obvious action of the symmetric group on the braid arrangement which permutes coordinates. And since the union of the hyperplanes H_{ij} contains all the points fixed under the action of a nontrivial permutation, the action on the complement is free. If we quotient by this action, the effect is to remove the labels from the points in the plane and the fundamental group of the quotient is the braid group. This relationship is captured by the fact that there is a natural epimorphism from the braid group to the symmetric group (where the image of a braid is the way it permutes its strands) and its kernel is the pure braid group.

The symmetric group action on the braid arrangement essentially changes the basepoint in the hyperplane complement and permutes the labels on the points in the plane. For each such basepoint there is a set of swing generators but recall that there is no natural isomorphism between the fundamental group of a connected space at one basepoint and its fundamental group at another. To create an isomorphsim one selects a path from the one to the other and then conjugates by this path. In our case such a path projects to a loop in the quotient by the symmetric group action and thus represents an element of the braid group. In particular, the resulting isomorphism between the fundamental groups is induced by an inner automorphism of the braid group which descends to an automorphism of its pure braid subgroup. By Corollary 1.4 this automorphism of P_n does not alter the BNS-invariant or its complement.

It does, however, change the standard generating set. If we keep track of the motion of the points in the plane dictated by the path between the basepoints, we find that the straight line segment between p_i and p_j used to define S_{ij} becomes an embedded arc between the images of these points that is typically very convoluted. In other words, the image of the original swing generator S_{ij} is a nonstandard generator where the points p_i and p_j travel along the twisted embedded arc from either end until they are very close, they then rotate fully around each other clockwise and then they return the way they came. Despite the fact that the image of a standard generator is no longer standard, it is true that the new nonstandard generator is conjugate in P_n to the standard generator between these two points. In particular, for any character χ, the χ-value of a standard generator S_{ij} is equal to the χ-value of the standard generator between the images of p_i and p_j under this automorphism of P_n.

We conclude this section with a discussion of epimorphisms between pure braid groups.

Definition 2.7 (natural projections). For every subset $A \subset \{1, 2, \ldots, n\}$ of size k there is a natural projecting epimorphism $\phi_A \colon P_n \twoheadrightarrow P_k$ which can be described topologically as "forgetting" what happens to the points not in A. Algebraically ϕ_A sends a standard generator S_{ij} to zero unless both endpoints belong to A. This produces $\binom{n}{k}$ epimorphisms from P_n onto P_k which are all distinct. The situations with $k = 3$ or $k = 4$ are particularly important here and we denote these maps by ϕ_{ijk} and ϕ_{ijkl} where the subscripts indicate the points contained in A.

The fact that the complete graph on 4 vertices is planar leads to a nice presentation for P_4 and a surprising projection from P_4 onto P_3.

Definition 2.8 (planar presentation of P_4). If we pick a basepoint for the braid arrangement that corresponds to the configuration of points shown in Figure 2, then the six straight segments connecting them pairwise produce six swing generators that form a nonstandard generating set for P_4. We denote these a through f as indicated. The following is a presentation for P_4 in this generating set:

$$P_4 \cong \left\langle a, b, c, d, e, f \;\middle|\; \begin{array}{l} abc = bca = cab,\ ad = da \\ cde = dec = ecd,\ be = eb \\ bfd = fdb = dbf,\ cf = fc \end{array} \right\rangle$$

We call this the *planar presentation of* P_4. This presentation appears to be folklore: it is well-known to experts in the field but we cannot find a reference to it in the literature. Since it is straightforward to produce this presentation from one of the standard presentations, we omit the derivation.

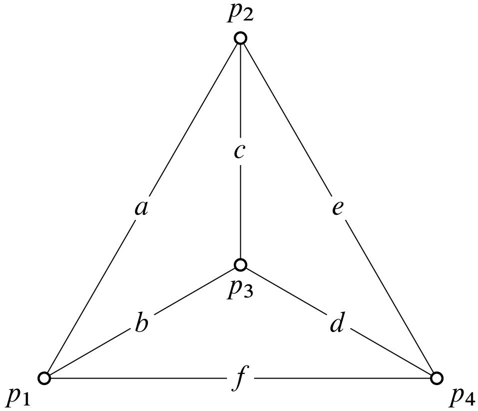

Figure 2. A labeled planar embedding of K_4.

Lemma 2.9 (an unusual map). *There is a morphism* $\rho\colon P_4 \twoheadrightarrow P_3$ *which sends the pairs of generators representing disjoint edges in the planar presentation of* P_4 *to the same standard generator of* P_3. *Concretely, the function that sends both* a *and* d *to* $a = S_{12}$, *both* b *and* e *to* $b = S_{13}$ *and both* c *and* f *to* $c = S_{23}$ *extends to such an epimorphism.*

Proof. Since the image of this function is a generating set, the only thing to check is that images of the planar generators satisfy the planar relations. This is clear since $a^2 = a^2$, $b^2 = b^2$, $c^2 = c^2$ and $abc = bca = cab$ in P_3. $\square$

3. Graphs

In this section we record a few miscellaneous remarks related to graphs that we use in the proof of the main result. The first is the definition of an auxilary graph that organizes information about a character, the second is an elementary result from linear algebra, and the third is a structural result about graphs that avoid a particular condition.

Definition 3.1 (graph of a character). For each character χ of P_n we construct a graph K_χ that we call the *graph of* χ. It is a subgraph of the complete graph K_n, it contains all vertices v_i with i in $\{1, 2, \ldots, n\}$ and it contains the edge e_{ij} from v_i to v_j if and only if the standard generator S_{ij} survives under χ. When working with examples, it is convenient to add labels to the edges of K_χ which record the χ-values of the corresponding standard generator. For example, the labeled graph shown in Figure 3 comes from a character which sends S_{24} to 0 and S_{13} to 2. For any set $A \subset [n]$, the χ-value of S_A can be recovered from K_χ by adding up the labels on the edges with both endpoints in A. Thus the character whose graph is shown in Figure 3 sends S_{124} to -1, S_{123} to 0 and S_{1234} to -3.

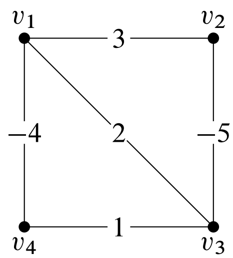

Figure 3. The graph of the character.

Lemma 3.2 (triple sums). *Let χ be a character for P_4. If the four values $\chi(S_{123})$, $\chi(S_{124})$, $\chi(S_{134})$ and $\chi(S_{234})$ are all zero, then $\chi(S_{12}) = \chi(S_{34})$, $\chi(S_{13}) = \chi(S_{24})$, $\chi(S_{14}) = \chi(S_{23})$, and $\chi(S_{1234}) = 0$.*

Proof. The proof is elementary linear algebra. If we expand the four given values as sums over the edges of K_χ and then add all four equations together, we find that twice the sum over all six edges is zero. If we then add two of the triangle equations and subtract the sum of six edges, we find that difference in the values of χ on a pair of disjoint edges is zero. In other words, their values are equal. This completes the proof. $\square$

And finally, we consider a condition on a graph that arises in Section 5 and which has quite strong structural implications.

Lemma 3.3 (star or small). *Let Γ be a graph with no isolated vertices. If Γ does not contain an edge disjoint from two other edges, then all edges of Γ have an endpoint in common, or they collectively have at most 4 endpoints. In other words, Γ is a star or a subgraph of K_4.*

Proof. If Γ has a vertex v of valence more than 3, then edges ending at v are its only edges. Otherwise, the additional edge is disjoint from at least two of the edges with v as an endpoint, contradicting our assumption. If Γ has a vertex of valence 3, then these four vertices are the only vertices of Γ. Otherwise, there is an edge with only one endpoint in this set, it must end at one of the vertices other than v (since the edges ending at v are already accounted for) and thus it avoids two of the edges with v as an endpoint, contradiction. Finally, if Γ only has vertices of valence 1 and 2, it is a collection of disjoint paths and cycles. If there is a cycle, then there can be no other components and the cycle must have length at most 4. If there are multiple paths, there can only be two and they must both consists of a single edge. If there is only one path, it can have at most 3 edges. This completes the proof since a cycle of length at most 4, two paths of length 1 and a single path of length 3 are all subgraphs of K_4. □

4. Characters in the complement

In this section we recall the BNS-invariants for P_2 and P_3 and then use the various epimorphisms between pure braid groups to produce a series of circles in the complement of $\Sigma^1(P_n)$ that we call P_3-circles and P_4-circles. Since P_2 is abelian, its invariant is trivial to compute by Corollary 1.10.

Lemma 4.1 (2 points). *The group P_2 is isomorphic to $\mathbb{Z}$, its character sphere is $\mathbb{S}^0$, the set $\Sigma^1(P_2)^c$ is empty and $\Sigma^1(P_2)$ includes both points.*

The group P_3 is only slightly more complicated.

Lemma 4.2 (3 points). *The group P_3 is isomorphic to $\mathbb{F}_2 \times \mathbb{Z}$, its character sphere is $\mathbb{S}^2$, the set $\Sigma^1(P_3)^c$ is the equatorial circle defined by $\chi(\Delta) = 0$, where Δ is the generator of the center, and $\Sigma^1(P_3)$ is the complement of this circle.*

Proof. One presentation for P_3 is $\langle a, b, c \mid abc = bca = cab \rangle$ where $a = S_{12}$, $b = S_{13}$ and $c = S_{23}$. If we add $d = \Delta = S_{123}$ as a generator and use the equation $abc = d$ to eliminate c we obtain the following alternative presentation: $P_3 \cong \langle a, b, d \mid ad = da,\ bd = db \rangle$, from which it is clear that $P_3 \cong \mathbb{F}_2 \times \mathbb{Z}$ with the free group $\mathbb{F}_2$ generated by a and b and the central $\mathbb{Z}$ generated by $d = \Delta$. By Corollary 1.10 characters in $\Sigma^1(P_3)^c$ must send Δ to zero. On the other hand, those which do send Δ to zero are really characters of $\mathbb{F}_2$ and it is well-known that the BNS-invariant for a free group is empty. Thus $[\chi] \in \Sigma^1(P_3)^c$ if and only if $\chi(\Delta) = 0$. □

It is the circle of characters that forms the complement of $\Sigma^1(P_3)$ which produces multiple circles in the complement of $\Sigma^1(P_n)$ for $n > 3$.

Definition 4.3 (P_3-circles and P_4-circles). We say that χ is part of a P_3-*circle* if there exists a natural projection map ϕ_{ijk} (described in Definition 2.7) and a character ψ where $[\psi] \in \Sigma^1(P_3)^c$, such that $\chi = \psi \circ \phi_{ijk}$:

$$P_n \xrightarrow{\phi_{ijk}} P_3 \xrightarrow{\psi} \mathbb{R}.$$

More concretely, χ is part of a P_3-circle if and only if all the endpoints of edges in K_χ belong to a three element subset $\{v_i, v_j, v_k\}$ and the value of $\chi(S_{ijk})$ is zero.

In a similar fashion we say that χ is part of a P_4-*circle* if there exists a triple of maps:

$$P_n \xrightarrow{\phi_{ijkl}} P_4 \xrightarrow{\rho} P_3 \xrightarrow{\psi} \mathbb{R}$$

whose composition is χ where ϕ_{ijkl} is one of the natural projection maps, ρ is the unusual map described in Lemma 2.9 and $[\psi] \in \Sigma^1(P_3)^c$. More concretely, χ is part of a P_4-circle if and only if all the endpoints of edges in K_χ belong to a four element subset $\{v_i, v_j, v_k, v_l\}$, the equations $\chi(S_{ij}) = \chi(S_{kl})$, $\chi(S_{ik}) = \chi(S_{jl})$, $\chi(S_{il}) = \chi(S_{jk})$ hold and the sum of these three shared values is zero.

Using Lemma 1.3 we immediately conclude the following:

Theorem 4.4 (characters in the complement). *Let χ be a character of P_n. If χ is a part of a P_3-circle or a P_4-circle, then $[\chi] \in \Sigma^1(P_n)^c$. This produces $\binom{n}{3} + \binom{n}{4}$ circles in the complement.*

Proof. The definitions of P_3-circles and P_4-circles ensure that Lemma 1.3 may be applied to χ to complete the proof. The second assertion comes from the number of natural projections onto 3 points plus the number of natural projections onto 4 points. $\square$

5. Characters in the invariant

In this final section we show that every character of P_n that is not part of a P_3-circle or a P_4-circle is in $\Sigma^1(P_n)$. We begin with a series of lemmas which follow, directly or indirectly, from Lemma 1.9. Recall that we use Δ to denote the element S_A with $A = \{1, 2, \ldots, n\}$ which generates the center of P_n.

Lemma 5.1 (zero sum). *If χ is a character of P_n and $\chi(\Delta)$ is not zero, then $[\chi] \in \Sigma^1(P_n)$.*

Proof. Since Δ is central in P_n, this follows from Corollary 1.10. □

Lemma 5.2 (disjoint triple). *If χ is a character of P_n and K_χ contains three pairwise disjoint edges, then $[\chi] \in \Sigma^1(P_n)$.*

Proof. First permute the points so that e_{12}, e_{34} and e_{56} are edges in K_χ. Then let $J = \{S_{12}, S_{34}, S_{56}\}$ and let I be the full standard generating set for this arrangement. The graph $C(J)$ is a triangle and J dominates I because every standard generator commutes with at least one element in J. Lemma 1.9 completes the proof. □

Lemma 5.3 (disjoint from a pair). *If χ is a character of P_n and K_χ contains an edge disjoint from two other edges, then $[\chi] \in \Sigma^1(P_n)$.*

Proof. If all three edges are disjoint then Lemma 5.2 applies. Otherwise, permute the points so that e_{12}, e_{34} and e_{45} are edges in K_χ. Then let $J = \{S_{12}, S_{34}, S_{45}\}$ and let I be a modification of the standard generating set for this arrangement where S_{14} and S_{24} are removed and S_{145} and S_{245} are added in their place. This remains a generating set because $S_{145} = S_{14}S_{15}S_{45}$ so that S_{14} can be recovered from the other three, and likewise, $S_{245} = S_{24}S_{25}S_{45}$ so S_{24} can be recovered. The graph $C(J)$ is connected since both S_{34} and S_{45} commute with S_{12}. Since every standard generator (with the exception of S_{14} and S_{24}) commutes with some element in J, and S_{145} and S_{245} commute with S_{45}, J dominates I. Lemma 1.9 completes the proof. □

At this point, the Star-or-Small Lemma, Lemma 3.3, implies that our search for characters in $\Sigma^1(P_n)^c$ can be restricted to those whose graph is a star or a subgraph of K_4, plus possibly some isolated vertices.

Lemma 5.4 (stars). *If χ is a character of P_n and the edges of K_χ form a star with at least 3 edges, then $[\chi] \in \Sigma^1(P_n)$.*

Proof. If $\chi(\Delta)$ is not zero, then $[\chi] \in \Sigma^1(P_n)$ by Lemma 5.1. Otherwise, permute the points so that v_1, v_2 and v_3 are leafs of K_χ and v_4 is the vertex all edges have in common. Then let I be the standard generators for this arrangement and let J consist of the six elements S_{14}, S_{24}, S_{34}, S_{A_1}, S_{A_2}, and S_{A_3} where A_i is the set

$\{1, 2, \ldots, n\}$ with i removed. Because $\chi(\Delta) = 0$ and v_1, v_2 and v_3 are leaves of K_χ, we have that

$$\chi(S_{A_i}) = \chi(\Delta) - \chi(S_{i4}) = -\chi(S_{i4}) \neq 0$$

for $i \in \{1, 2, 3\}$. In particular, all of J survives under χ. Next since $\{j, 4\} \subset A_i$ so long as i and j are distinct elements in $\{1, 2, 3\}$, we have that S_{A_i} commutes with S_{j4} in these situations. As a consequence, the graph $C(J)$ is connected by a hexagon of edges. Finally, J dominates I since every standard generator avoids one of the first three points and thus commutes with one of the elements S_{A_i}. Lemma 1.9 completes the proof. □

Lemma 5.5 (disjoint leaves). *Let χ be a character of P_n. If K_χ contains two vertices of valence* 1 *and the unique edges that end at these vertices are disjoint, then* $[\chi] \in \Sigma^1(P_n)$.

Proof. If $\chi(\Delta)$ is not zero, then $[\chi] \in \Sigma^1(P_n)$ by Lemma 5.1. Otherwise, permute the points so that v_1 and v_3 are leaves and e_{12} and e_{34} are in K_χ. Then let I be the standard generators for this arrangement and let J consist of the five elements S_{12}, S_{34}, S_{123}, S_{A_1} and S_{A_3} where A_i is the set $\{1, 2, \ldots, n\}$ with i removed. Because $\chi(\Delta) = 0$ and v_1 and v_3 are leaves of K_χ, we have that $\chi(S_{A_1}) = -\chi(S_{12})$, $\chi(S_{A_3}) = -\chi(S_{34})$, and $\chi(S_{123}) = \chi(S_{12})$. In particular, all of J survives under χ. The graph $C(J)$ is connected since both S_{A_3} and S_{123} commutes with S_{12} which commutes with S_{34} which commutes with S_{A_1}. Finally, the set J dominates I because the only standard generator which does not commute with S_{A_1} or S_{A_3} is S_{13} and it commutes with S_{123}. Lemma 1.9 completes the proof. □

Lemma 5.6 (disjoint edges and one triangle). *Let χ be a character of P_n. If K_χ contains a pair of disjoint edges and three of these endpoints form a triangle whose χ-value is not zero, then* $[\chi] \in \Sigma^1(P_n)$.

Proof. Permute the points so that e_{12} and e_{34} are edges in K_χ and $\chi(S_{123})$ is not zero. Then let $J = \{S_{12}, S_{123}, S_{34}\}$ and let I be the standard generating set for this arrangement with S_{14} and S_{24} removed and S_{134} and S_{234} added in their place. The set I still generates P_n since $S_{134} = S_{13}S_{14}S_{34}$ and $S_{234} = S_{23}S_{24}S_{34}$ so S_{14} and S_{24} can be recovered from the ones that remain. The graph $C(J)$ is connected since S_{12} commutes with both S_{123} and S_{34}. Every standard generator with an endpoint outside the set $\{1, 2, 3, 4\}$ commutes with either S_{12} or S_{34}. Of the six standard generators with both endpoints in this set, two are not in I, three commute with S_{123}, and S_{34} commutes with itself. Finally, the added elements

S_{134} and S_{234} both commute with S_{34} so J dominates I. Lemma 1.9 completes the proof. □

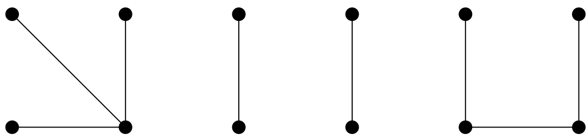

Figure 4. Three subgraphs of K_4.

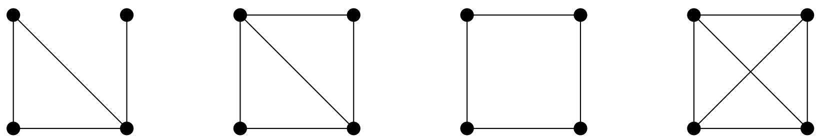

Figure 5. Four subgraphs of K_4.

These lemmas combine to prove the following.

Theorem 5.7 (Characters in the invariant). *Let χ be a character of P_n. If $[\chi]$ is not part of a P_3-circle or a P_4-circle then $[\chi] \in \Sigma^1(P_n)$.*

Proof. Let Γ be the graph K_χ with isolated vertices removed. By Lemma 5.3, Lemma 5.4, and Lemma 3.3, $[\chi]$ is in $\Sigma^1(P_n)$ unless Γ has at most 4 vertices. By Lemma 5.1, $[\chi] \in \Sigma^1(P_n)$ unless the sum of the edge weights is zero. So assume that Γ has at most 4 vertices, none of them isolated and the sum of the edge weights is zero. There are no 2 vertex graphs satisfying these conditions and the only 3 vertex graphs remaining are those which represent characters in P_3-circles. Thus we may also assume that Γ has exactly 4 vertices. Up to isomorphism there are precisely seven such graphs and they are shown in Figures 4 and 5. If Γ is isomorphic to one of three graphs in Figure 4, then $[\chi] \in \Sigma^1(P_n)$ by Lemma 5.4 or Lemma 5.5. Finally, the four graphs in Figure 5 all have disjoint edges. If any triple of vertices have edges whose χ-values have a non-zero sum, then $[\chi] \in \Sigma^1(P_n)$ by Lemma 5.6. The only remaining case is where all such triples sum to zero. By Lemma 3.2 this means that disjoint edges are assigned equal values. This rules out the two graphs on the left of Figure 5 and reduces the other two to graphs representing characters in P_4-circles. And this completes the proof. □

Theorem 4.4 and Theorem 5.7 prove Theorem A.

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