CHAPTER 1

Coxeter groups

Current Version
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Now that the flavor of the subject is clear, let's start back at the beginning in a slightly more general context. Let W be a finite group of isometries of $\mathbb{R}^n$ generated by reflections. The fact that W is finite is its most important property and this imposes enormous restrictions on the structure of the group W. Once the structure these finite reflection groups have been classified (and shown to be finite Coxeter groups) the discusssion will broaden once again to the class of all Coxeter groups.

1. Reflections and root systems

Definition 1.1 (Reflections). Each vector $\alpha \in \mathbb{R}^n$ determines a line $\mathbb{R}\alpha$ and a reflection s_α which acts on $\mathbb{R}^n$ by fixing the vectors perpendicular to α and reversing the line $\mathbb{R}\alpha$. Algebraically, if $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, then $s_\alpha(\beta)$ is found by subtracting off twice the projection of β onto α, in other words,$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}\alpha$. See Figure 1.

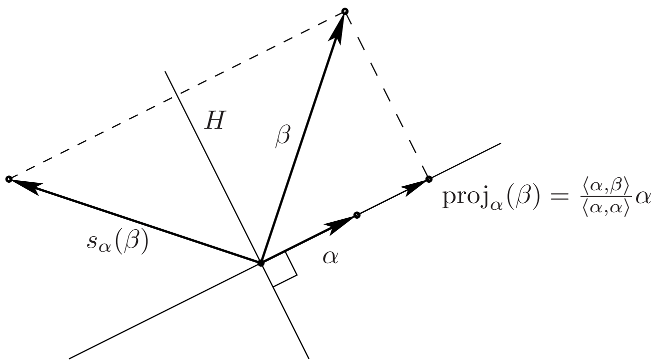

Figure 1. Reflecting β through the hyperplane H perpendicular to α.

Proposition 1.2 (Reflections under restriction). *A linear transformation $s : \mathbb{R}^n \to \mathbb{R}^n$ is a reflection if and only if there is an orthogonal basis of $\mathbb{R}^n$ such that the matrix of s relative to this basis is a diagonal matrix with all 1s on the diagonal except for one -1. As a consequence, if $s = s_\alpha$ is a reflection of $\mathbb{R}^n$ and V is any subspace of $\mathbb{R}^n$ which contains the line $\mathbb{R}\alpha$, then s restricted to V is a reflection of V.*

PROOF. If s is a reflection the picking an orthogonal basis for $\mathbb{R}^n$ which includes α creates a matrix of the proper form, and conversely, it is clear that an orthogonal basis which produces a matrix of the proper form describes a reflection since the span of the special basis element is flipped by s, the span of the remaining basis elements is fixed by s, and these are perpendicular. For the second assertion simply pick an orthogonal basis for V containing α and then extend this to an orthogonal basis for $\mathbb{R}^n$. The matrix of the restriction of s to V is now seen to be of the right form. $\square$

DEFINITION 1.3 (Root system). A *root system* is a finite collection Ω of vectors in $\mathbb{R}^n$ such that (1) $\mathbb{R}\alpha \cap \Omega = \{\alpha, -\alpha\}$ and (2) $s_\alpha \Omega = \Omega$ for all $\alpha \in \Omega$.

One tool which helps facilitate several of the algebraic proofs is the notion of a total ordering of a real vector space which is compatible with the vector space structure.

DEFINITION 1.4 (Total orderings). Let V be a real vector space. For our purposes, a *total ordering* (define total orderings, lexicographic ordering, etc)

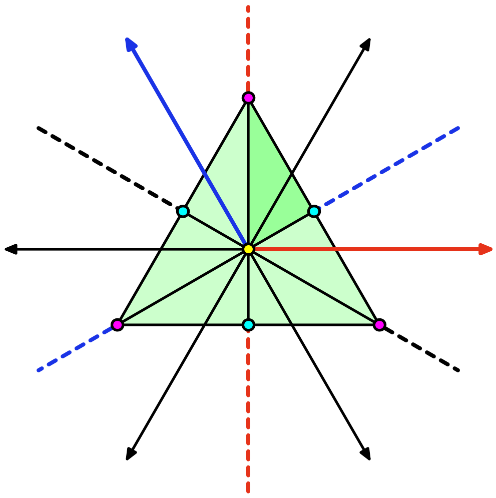

FIGURE 2. The barycentric subdivision of an equilateral triangle with a fundamental chamber selected, its 3 reflecting hyperplanes and its 6 root vectors.

DEFINITION 1.5 (Positive system). Let Φ be a root system. A *positive system* Φ^+ is a selection of a positve root along each line $\mathbb{R}\alpha$ which is consistent with some total ordering. Given a set of vectors such as Φ^+, its *positive cone* is the set of nonnegative linear combinations of the vectors in Φ^+. Because the set of vectors which are nonnegative in the total ordering is closed under nonnegative scalar multiples and vector addition, the positive cone of any positive system does not include any vectors which are less than the zero vector.

DEFINITION 1.6 (Simple system). Let Φ be a root system and let Φ^+ be one of its positive systems. A *simple system* Δ for Φ^+ is a minimal subset of Φ^+ which generates the same positive cone as Φ^+.

Lemma 1.7 (Angles in simple systems). *Let Φ be a root system, let Φ^+ be a positive system and let Δ be a simple system for Φ^+. For all $a, b \in \Delta$, $\langle a, b\rangle \leq 0$. In other words, the vectors in Δ are "spread out".*

Proof. The claim is easy to check for 2-dimensional (i.e. dihedral) root systems and the general case quickly reduces to the 2-dimensional case. More specifically, look at the roots in the plane spanned by a and b. By assumption there are only finitely many such roots and because the corresponding reflections send this plane to itself, they form a root system of their own. $\square$

Proposition 1.8 (Simple systems are bases). *Every simple system is a basis of $\mathbb{R}^n$. As a consequence, any simple system has n elements.*

Proof. Let Δ be a simple system in the root system Φ. It's already clear that the vectors in Δ span $\mathbb{R}^n$ so the only question is whether

If $|\Delta|$ Suppose $\square$

2. Cayley graphs and permutahedra

Definition 2.1 (Cayley graphs). If G is a group generated by a set A, then the *right Cayley graph* $\text{Cayley}(G, A)$ is the labeled directed graph whose vertices are labeled by elements of G and whose edges are in one-to-one correspondence with the set $G \times A$ where the edge corresponding to the ordered pair (g, a) starts at the vertex g, ends at the vertex labeled $g \cdot a$ and has an edge label of a. The *left Cayley graph* is defined similarly except that the edge corresponding to (g, a) ends instead at the vertex $a \cdot g$.

Lemma 2.2 (Left actions on right Cayley graphs). *If G is a group generated by a set A, then G acts on the left on the right Cayley graph $\text{Cayley}(G, A)$, and G acts on the right on the left Cayley graph.*

Proof. This result follows from associativity of the multiplication. Let e be an edge of the right Cayley graph labeled by $a \in A$ connecting g to $g \cdot a$. If h is any element of G, then the left action of h on the Cayley graph is the map denoted $h\cdot$ which sends a vertex labeled g to $h \cdot g$ and an edge such as e to the edge labeled by a connecting $h \cdot g$ and $h \cdot (g \cdot a)$. Since $h \cdot (g \cdot a) = (h \cdot g) \cdot a$ there really is an edge labeled by a connecting these vertices. $\square$

Almost as important as the lemma itself is the observation that there does not exist a left action of G on the left Cayley graph when G is nonabelian. This is because there is at least one instance where $h \cdot (a \cdot g)$ and $a \cdot (h \cdot g)$ describe different elements of G. Thus, the switch between left and right is essential to the assertion.

Definition 2.3 (Permutohedra). (define)

Example 2.4. For the isometry group of the n-simplex, i.e. the finite reflection group of type A_{n-1}, the resulting shape (using the standard coordinate system) is the convex hull of the $n!$ vectors in $\mathbb{R}^n$ with coordinates 1 through n in some order. In the convex hull, the 1-cells connect those vertices which differ by a single transposition which switches two positions containing successive integers. Thus $(2, 3, 5, 1, 4)$ is connected by edges with $(1, 3, 5, 2, 4)$, $(3, 2, 5, 1, 4)$, $(2, 4, 5, 1, 3)$ and $(2, 3, 4, 1, 5)$.

Definition 2.5 (Minkowski sum). Let P and Q be two convex polytopes situated in a common vector space such as $\mathbb{R}^n$. The *Minkowski sum* of P and Q is defined as follows.

$$P + Q = \{\vec{u} + \vec{v} \mid \vec{u} \in P \text{ and } \vec{v} \in Q\}$$

It is straightforward to check that $P + Q$ is also a convex polytope in R^n, its faces are products of particular faces of P and Q, and that this operation is commutative and associative since vector addition has these properties.

Proposition 2.6 (Sum of the root system). *If W is a finite reflection group with root system Φ, then the W-permutahedron is a Minkowski sum of the root vectors in Φ.*

Proof. It is easy to check that the Minkowski sum of the root vectors has all of the properties defining the W-permutahedron. It's clearly a convex polytope which is invariant under the action of W, etc. (finish this) □

Proposition 2.7. *Let W be a finite reflection group with root system Φ and let P be its W-permutahedron. If Δ is a simple system in Φ, then, with proper labeling, the 1-skeleton of P can be identified with the right Cayley graph of W with respect to the reflections corresponding to Δ.*

The notions of a positive system and a simple systems can be reinterpreted geometrically.

Theorem 2.8. *Let W be a finite reflection group with root system Φ, let H be its hyperplane arrangement, and let P be its W-permutahedron. There are natural bijections between the following sets:*

1. *elements of W,*
2. *chambers of H,*
3. *vertices of P,*
4. *small neighborhoods of the vertices of P,*
5. *the sets of directed edges leaving each vertex of P,*
6. *simple systems of Φ,*
7. *positive system of Φ,*
8. *the positive cones associated to each positive system.*

Definition 2.9 (Length). If we identify the 1-skeleton of the W-permutahedron with the right Cayley graph of W.

3. Finite Coxeter groups

Definition 3.1 (Coxeter groups).

Lemma 3.2 (Reflection $\Rightarrow$ Coxeter). *Every finite reflection group is a finite Coxeter group. In particular, if W is a finite reflection group and S is a set of simple reflections in W, then the Coxeter presentation generated by S whose relations record the product of each pair of elements in S is, in fact, a presentation of the group W.*

Proof. Viewing the W-permutahedron as a cell complex, it is clear that it's 2-skeleton is simply-connected (since it is the 2-skeleton of a topological ball). As noted above the 1-skeleton is isomorphic to the Cayley graph of W with respect to S and each 2-cell has a boundary cycle labeled by $(ab)^m$ for some $a, b \in S$ with m being their order. □

Ever since Moussong's dissertation it has been known that all Coxeter groups are CAT(0) groups. The situation for Artin groups, however, remains far from clear.

Coxeter groups first arose in the classification of finite groups acting on Euclidean space which are generated by reflections (i.e. isometries fixing a codimension 1 hyperplane). In fact, the collection of finite Coxeter groups is the same as the collection of finite reflection groups. The well-known classification of irreducible Coxeter groups divides them into type A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, H_3, H_4 and $I_2(m)$ $(m \geq 2)$. The corresponding diagrams (using the alternative convention) are shown in Figure 3. For more details see some of the standard references for Coxeter groups such as Bourbaki [**2**], Humphreys [**12**] or Kane [**13**].

THEOREM 3.3 (Finite Coxeter groups). *Every finite Coxeter group is a finite reflection group. In particular, every finite Coxeter group is either the isometry group of a regular polytope, or it is described by one of the Dynkin diagrams shown in Figure 3.*

PROOF. (how? I need to show that not being positive definite implies infinite group) □

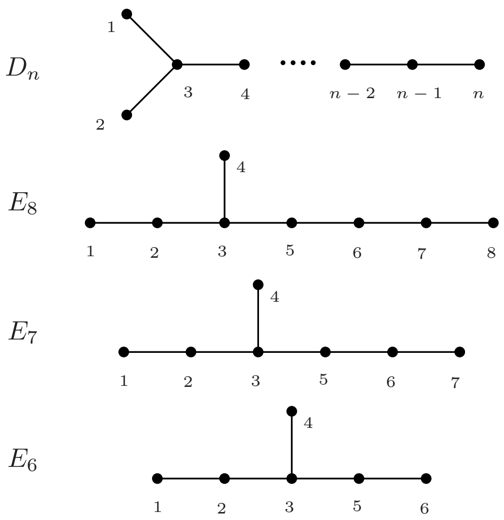

FIGURE 3. Additional diagrams for the irreducible finite Coxeter groups.

The classification goes as follows (1) no infinite edge labels (2) no loops so tree, (3) if not a regular polytope then there exists a branch point

(4) no two branch points and (5) no high labels with branch point so small type tripod. At this point calculate determinant to see that $\frac{1}{I}+\frac{1}{J}+\frac{1}{K} > 1$. Thus, either two 2's (D_n) or $(2,3,3)$, $(2,3,4)$, or $(2,3,5)$ which are E_6, E_7, and E_8 respectively.

Include constructions of D_n and E_8.

4. Numerology

There is quite a bit of numerology surrounding the finite Coxeter groups. For example, let W be a finite Coxeter group acting on $\mathbb{R}^n$ by reflections with root system Φ and let S be the elements in W corresponding to a simple system in Φ. To fix notation, let N be the size of Φ^+, i.e. the number of reflections and let n be the size of Δ, i.e. the number of basic reflections.

The product of the elements in S is called a *Coxeter element*. Coxeter elements are not unique since changing the order changes the result, but all Coxeter elements have many properties in common. First, they are all conjugate to each other and thus all have the same order. This order, h, is called the *Coxeter number*. Let w be a particular Coxeter element of W. There is a special 2-plane in $\mathbb{R}^n$ which is invariant under the action of w. Moreover, the action of w on this plane is rotation by $2\pi/h$. Because w has order h, all of the eigenvalues of the matrix M_w representing the action of w are h^{th} roots of unity. In particular, is $\zeta = e^{2\pi i/h}$ is a primitive h^{th} roots of unity then all of the eigenvalues of M_w are powers of ζ. The list of these powers are called the *exponents of* W. Perhaps surprisingly, the numbers in this list are distinct. When listed in order they are called $e_1, e_2, \ldots, e_n$. The *degrees of* W, technically, are the degrees of a basis of W-invariant polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. For our purposes, it suffices to remark that $d_i = e_i + 1$. Thus, for example, the exponents of H_3 are $1, 5, 9$ and its degrees are $2, 6, 10$. The basic data for the finite reflection groups is shown in Figure 4.

Some of the equations which are known to hold between these basic data are as follows. (say more)

5. General Coxeter groups

(this needs lots of work. Current version simply pulled from elsewhere)

Definition 5.1 (Artin groups and Coxeter groups). Let Γ be a finite graph with edges labeled by integers greater than 1, and let $\langle a, b\rangle^n$ denote the length n prefix of $(ab)^n$. The *Artin group* A_Γ is the group generated by a set in one-to-one correspondence with the vertices of Γ with a relation of the form $\langle a, b\rangle^n = \langle b, a\rangle^n$ whenever a and b correspond to vertices joined by an edge labeled n. The *Coxeter group* W_Γ is the Artin group A_Γ modulo the additional relations $a^2 = 1$ for each generator a. There is also an alternative convention for associating diagrams with Coxeter groups and Artin groups which is derived from a consideration of the finite Coxeter groups. In that case, the graph Γ (using the above convention) is always a complete graph with most of the edges labeled 2 or 3. The alternate convention simplifies the diagram by removing all edges labeled 2 and leaving the label implicit for the edges labeled 3.

Example 5.2. Let Γ denote the labeled graph shown in Figure 7. The presentation of the Artin group A_Γ is $\langle a, b, c|\ aba = bab, ac = ca, bcbc = cbcb\rangle$ and the presentation $\left\langle a, b, c | aba = bab, ac = ca, bcbc = cbcb, a^2 = b^2 = c^2 = 1\right\rangle$ defines the Coxeter group W_Γ.

6. Davis complex

This section shows how to build a nice complex out of permutahedra on which an arbitrary Coxeter group acts by isometries. The basic idea is that the Cayley

X_n	**h**	**2N**	**N**	**Exponents**	**Degrees**	**Size**
A_2	3	6	3	1,2	2,3	6
B_2	4	8	4	1,3	2,4	8
A_3	4	12	6	1,2,3	2,3,4	24
B_3	6	18	9	1,3,5	2,4,6	48
H_3	10	30	15	1,5,9	2,6,10	120
A_4	5	20	10	1,2,3,4	2,3,4,5	120
D_4	6	24	12	1,3,5,3	2,4,6,4	192
B_4	8	32	16	1,3,5,7	2,4,6,8	384
F_4	12	48	24	1,5,7,11	2,6,8,12	1,152
H_4	30	120	60	1,11,19,29	2,12,20,30	14,400
A_5	6	30	15	1,2,3,4,5	2,3,4,5,6	720
D_5	8	40	20	1,3,5,7,4	2,4,6,8,5	1,920
B_5	10	50	25	1,3,5,7,9	2,4,6,8,10	3,840
A_6	7	42	21	1,2,3,4,5,6	2,3,4,5,6,7	5,040
D_6	10	60	30	1,3,5,7,9,5	2,4,6,8,10,6	23,040
B_6	12	72	36	1,3,5,7,9,11	2,4,6,8,10,12	46,080
E_6	12	72	36	1,4,5,7,8,11	2,5,6,8,9,12	51,840
A_7	8	56	28	1,2,3,4,5,6,7	2,3,4,5,6,7,8	40,320
D_7	12	84	42	1,3,5,7,9,11,6	2,4,6,8,10,12,7	322,560
B_7	14	98	49	1,3,5,7,9,11,13	2,4,6,8,10,12,14	645,120
E_7	18	126	63	1,5,7,9,11,13,17	2,6,8,10,12,14,18	2,903,040
A_8	9	72	36	1,2,3,4,5,6,7,8	2,3,4,5,6,7,8,9	362,880
D_8	14	112	56	1,3,5,7,9,11,13,7	2,4,6,8,10,12,14,8	5,160,960
B_8	16	128	64	1,3,5,7,9,11,13,15	2,4,6,8,10,12,14,16	10,321,920
E_8	30	240	120	1,7,11,13,17,19,23,29	2,8,12,14,18,20,24,30	696,729,600
A_9	10	90	45	1,2,3,4,5,6,7,8,9	2,3,4,5,6,7,8,9,10	3,628,800
D_9	16	144	72	1,3,5,7,9,11,13,15,8	2,4,6,8,10,12,14,16,9	92,897,280
B_9	18	162	81	1,3,5,7,9,11,13,15,17	2,4,6,8,10,12,14,16,18	185,794,560

FIGURE 4. The basic data for the finite type Coxeter groups.

Type	**h**	**2N**	**N**
A_n	$n-1$	$n(n-1)$	$n(n-1)/2$
D_n	$2(n-1)$	$2n(n-1)$	$n(n-1)$
B_n	$2n$	$2n^2$	n^2

FIGURE 5. The basic data for the infinite families of finite Coxeter groups, Part I.

graph of an arbitrary Coxeter group with respect to its standard generating set contains the 1-skeleta of certain obvious permutahedra. In particular, if a subset of the standard generating set generate a finite Coxeter group W, then there the portion of the Cayley labeled by these generators form a disjoint union of 1-skeleta of W-permutahedra. We attach a W-permutahedron to each of these to produce the Davis complex. Since each permutahedron comes equipped with the metric of a Euclidean polytope and these metrics agree on overlaps, there is a well-defined intrinsic metric on the entire complex.

Type	**Exponents**	**Degrees**	**Size**
A_n	$1, 2, \ldots, n-1$	$2, 3, \ldots, n$	$n!$
D_n	$1, 3, \ldots, 2n-3, n-1$	$2, 4, \ldots, 2n-2, n$	$2^{n-1} \cdot n!$
B_n	$1, 3, \ldots, 2n-3, 2n-1$	$2, 4, \ldots, 2n-2, 2n$	$2^n \cdot n!$

FIGURE 6. The basic data for the infinite families of finite Coxeter groups, Part II.

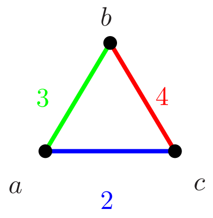

FIGURE 7. A labeled graph used to define a Coxeter group and an Artin group.