Introduction

Geometric group theory classifies groups by the nature of the spaces on which the groups act geometrically.

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Geometric group theory is a relatively young field, but it has deep roots in the study of groups from combinatorial and topological perspectives. For almost one hundred years combinatorial group theorists have viewed groups as essentially topological objects and they have used the topological invariants of combinatorial cell complexes to study their associated fundamental groups. Since the mid-1980s, spurred on by the seminal ideas of Jim Cannon and Misha Gromov, group theorists have paid increasing attention to the geometric structures these cell complexes can carry. Finitely generated groups are now also viewed as inherently metric objects.

The addition of a geometric perspective has been tremendously successful at solidifying previously disparate results, generating new questions for researchers to investigate, and enabling rapid progress on many fronts. An unfortunate corollary of this rapid expansion has been a separation between the background acquired by graduate students in their standard courses and the conceptual tools used by current researchers in the field. This book is my attempt to partially fill this gap.

Groups as Actions

One way to appreciate the naturalness of the geometric group theory approach is to take a step back and consider the way in which groups arise in mathematics more generally. Group theory comes from the study of symmetry, where a *symmetry* of an object P (or an equation, or a geometric configuration, or any other mathematical structure) is a non-trivial invertible map f from P to P that preserves the properties we wish to consider. The collection of all such maps, trivial or not, is clearly closed under function composition (automatically an associative operation), it includes the identity map, and it includes the inverses of these maps by definition. These *symmetry groups* are where the subject began. To this day, groups are often first introduced through a careful examination of the symmetry groups of specific geometric objects, such as regular n-gons, regular n-simplices, or the unit n-sphere. The resulting groups are the dihedral, symmetric, and orthogonal groups in our examples, or, if we restrict our attention to only those symmetries realizable as continuous motions inside $\mathbb{R}^2$, $\mathbb{R}^n$, or $\mathbb{R}^{n+1}$, respectively, we get the cyclic, alternating, and special orthogonal groups. Geometric objects, of course, are not the only mathematical structures that have symmetry groups. The symmetries of a vector space V form the *general linear group* $GL(V)$ and, more generally, the symmetries of any mathematical structure is called its *automorphism group*.

But if the abstractions pursued by twentieth century mathematicians have taught us anything, it is that mathematical structures should always be considered in conjunction with their structure-preserving homomorphisms and these maps also have symmetry groups! If $f : P \to Q$ is any structure-preserving homomorphism, then the collection of all invertible structure-preserving maps $g : P \to P$ such that $f \circ g = f$ form a group, as do the collection of all invertible structure-preserving maps $h : Q \to Q$ such that $h \circ f = f$. We can think of these groups as the *right* and *left stabilizer groups* of the map f, respectively. These types of groups also occur throughout mathematics. If $f : k \to K$ is a (necessarily injective) field homomorphism, for example, then its left stabilizer is better known as the *Galois group* of K over k. A second example, and one that is particularly important in our context, is when X is a path-connected topological space that has a universal cover $\widetilde{X}$ and $p : \widetilde{X} \to X$ is the natural covering projection. The right stabilizer of p is the *group of deck transformations* of p, and it happens to be isomorphic to the *fundamental group* of X.

In each of the situations described above, the group under consideration is acting on some mathematical object via structure-preserving maps. The structure of the object upon which the group is acting can then be used to extract detailed information about the group itself. In some sense, this is the main way that groups occur "in nature", as mathematicians like to say, and it is primarily through such actions, or *representations*, that groups are studied.

Finitely Presented Groups

Groups are investigated via representations as actions, but the type of representation varies with the type of group under consideration. For finite groups, group actions on finite sets (called *permutation representations*) or on vector spaces (known as *linear representations*) are highly effective and extensively used.[1] Geometric group theorists, on the other hand, focus their attention on groups that can be analyzed using actions on topological spaces—particularly cell complexes and metric spaces—and these often have infinitely many elements.

Infinite groups remain mysterious to many mathematics majors, since the groups encountered in a typical abstract algebra course are mostly finite. This is partly out of necessity: the main tools used to study infinite groups require more topology and geometry than can be presumed at that point. Moreover, when studying infinite groups, the algebraic structure often recedes into the background as topological, geometrical and logical considerations play a greater role.

Once infinite groups are under consideration, logical and informational issues immediately arise. Which infinite groups should be studied? If we are too inclusive in our scope, set theoretic issues could easily play a dominating role. On the other hand, the scope should be broad enough to include interesting examples, such as the fundamental groups of compact manifolds with or without boundary. One approach would be to limit our attention to precisely these groups. The obvious follow-up question is which groups are these? It turns out that this particular class of groups has several equivalent characterizations. Algebraically, they are the groups G that

[1] These types of representations have been particularly important in the classification of the finite simple groups. See Michael Aschbacher's book on finite group theory [**1**] for an excellent illustration of this approach and its benefits.

can be *finitely presented* in the following sense: (1) there exists some finite set of elements that generate all of G and (2) the relations that hold among the words in these *generators* can be derived from a finite list of basic rules or *relations*. Two other descriptions that describe the same class of groups are the fundamental groups of compact cell complexes and the fundamental groups of finite simplicial complexes. In other words, the following four collections of groups are identical.

$$\begin{aligned} \{ \text{ finitely presented groups } \} &= \{ \ \pi_1 \text{ of compact manifolds } \} \\ &= \{ \ \pi_1 \text{ of compact cell complexes} \} \\ &= \{ \ \pi_1 \text{ of finite simplicial complexes } \} \end{aligned}$$

This natural class of groups will be our primary focus, although it is sometimes convenient to consider groups that are finitely generated but not finitely presented, or even groups where no finite subset generates the whole group.

While it is certainly possible to develop the theory of finitely presented groups using the algebraic description with only a passing mention of topology and geometry, doing so makes many of the fundamental properties of infinite groups unnecessarily difficult to express and even harder to establish. As a geometric group theorist, I have tried instead to highlight the geometric and topological aspects as much as possible.

Scope and Prerequisites

As it has grown over the past twenty years, geometric group theory has developed strong connections with geometry, topology, analysis and logic and each of these facets is currently undergoing rapid development. It would be nearly impossible at this point to give a truly comprehensive introduction to geometric group theory in a single volume and the text you have before you is not intended as one.[2] I have tried instead to produce a book that thoroughly covers a cohesive subset of fundamental ideas, focusing on a selection of elementary and intermediate topics that I feel are absolutely essential. Such a selection is, of course, highly subjective. While I am confident about the centrality of the included topics, the reader should not infer that excluded ones are less important.

The foundational ideas in geometric group theory are fairly accessible and the required prerequisites are correspondingly minimal: the algebraic topology covered in Hatcher's book [**16**] is more than sufficient. In fact, if the reader is willing to take a few of the theorems listed in Appendix A on faith, the entire book can be understood after completing a course on fundamental groups and covering spaces.

[2] In fact, extensive volumes already exist or are nearing completion on several topics that are mentioned here only in passing. See the Epilogue for an extended discussion of these additional resources.

Structure of the Text

The structure of the text is relatively straightforward. After a prologue designed to whet the reader's appetite, there is one introductory chapter, two chapters that present the core philosophy behind geometric group theory, two chapters that examine the special role played by hyperbolic metrics, and two chapters that cover more advanced topics. Finally, there is an epilogue that tries to ease the transition into the research literature, and an appendix that reviews those aspects of basic algebraic topology that serve as a foundation for the subject.

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