NOTES FOR SEPTEMBER 29, 2004

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When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of the others.

Augustin-Louis Cauchy (1789-1857)

1. Limits and Continuity

Let's start with some definitions.

Definition 1.1 (Limit of a sequence)**.** Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers. The limit of the sequence (s_n) is ℓ if for every $\epsilon > 0$ there exists an $N > 0$ such that $n > N$ implies $|s_n - \ell| < \epsilon$.

The definition of a limit of a function is similar.

Definition 1.2 (Limit of a function)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The limit of the f as x approaches a is ℓ if for every $\epsilon > 0$ there exists an $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - \ell| < \epsilon$.

Why is this so hard for students to understand the first time they see it? One possibility is that this is one of their first exposures to dealing with multiple quantifiers. Quantifiers are tricky. Their order is extremely important. Another answer is that this business about taking a *deleted* neighborhood of a ruins the symmetry in the definition. The definition of continuity at a point doesn't suffer from this defect.

Definition 1.3 (Continuity at a point)**.** A function $f : \mathbb{R} \to \mathbb{R}$ (with $f(a) = b$) is *continuous at a* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - b| < \epsilon$. Other ways of writing this implication are:

- $x \in B_\delta(a)$ implies $f(x) \in B_\epsilon(b)$,
- $f(B_\delta(a)) \subset B_\epsilon(b)$, or
- $B_\delta(a) \subset f^{-1}(B_\epsilon(b))$.

Remark 1.4. Notice that continuity at a point is a very unintuitive concept! It is only the notion of a function which is continous on its entire domain which corresponds even remotely to our intuitive notion of continuity. Problem #21 in Sutherland illustrates this nicely.

With continuity over an interval we get such familiar theorems as the IVT and EVT.

Theorem 1.5 (Intermediate Value Theorem)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and d lies strictly between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.*

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Theorem 1.6 (Extreme Value Theorem)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ then $f([a, b])$ is bounded above and below and it attains its extreme values.*

2. Metric spaces

The general definition of a metric space is probably familiar to all of you.

(M1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
(M2) $d(x, y) = d(y, x)$.
(M3) $d(x, y) + d(y, z) \geq d(x, z)$.

A function which only fails the second half of M1 is called a *pseudometric*.

Example 2.1. Some examples are the usual metric on $\mathbb{R}^n$ (called d_2). There is also d_1, d_∞, d_0 and d_p. For functions there are L^1, L^2, L^p (as well as the uniform metric) and for sequences there are ℓ^1, ℓ^2 and ℓ^p as well. These will be explained in detail on Friday.

Definition 2.2 (Discrete and uniform metrics)**.** The *discrete metric* is $d(x, y) = 0$ or 1 depending on whether $x = y$ or $x \neq y$. This metric can turn an arbitrary set into a metric space. The *uniform metric* is a metric on the set of bounded functions $f : X \to \mathbb{R}$ defined by $d(f, g) = \sup_X |f(x) - g(x)|$. Again this works for any set X.

Note that the uniform metric is the one needed to cleanly state assertions about uniform convergence.

Definition 2.3 (L^p and ℓ^p)**.** Consider the collection of functions $f : [a, b] \to \mathbb{R}$ such that the following distance is well-defined.

$$d_p(f, g) = \left(\int_a^b |f(x) - g(x)|^p \, dx \right)^{1/p}$$

This is called the L^p metric. Alternatively, consider the collection of sequences $s : \mathbb{N} \to \mathbb{R}$ such that the following distance is well-defined.

$$d_p((s_n), (t_n)) = \left(\sum_{\mathbb{N}} |s_n - t_n|^p \right)^{1/p}$$

This is called the ℓ^p metric. A similar definition can be made on the collection of finite sequences. These are (sometimes) known as the ℓ^p metrics on $\mathbb{R}^n$.