NOTES FOR SEPTEMBER 27, 2004

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God made the integers, and all the rest is the work of man.
Leopold Kronecker (1823-1891)

1. Set theory

You should be familiar with the basics of set theory up to and including the Axiom of Choice. Here is a brief review of some of the highlights.

Remark 1.1 (Building the number systems)**.** Should one choose to do so, the integers can be constructed out of the naturals, the rationals out of the integers, and the complex numbers out of the reals by fairly elementary algebraic constructions. The construction of the reals from the rationals is another matter entirely. The two most common methods are by means of Dedekind cuts and by metric space completion. The latter is the method outlined in the appendix.

Recall that an ordered set is said to be *well-ordered* if every nonempty subset has a least element.

Axiom 1.2 (Well-ordering axiom)**.** *The natural numbers with their usual ordering is a well-ordered set.*

This axiom is logically equivalent to the principle of induction. There is also a strong version which extends this property to all sets.

Axiom 1.3 (Well-ordering principle)**.** *Every set has an ordering which is well-ordered.*

This stronger version is logically equivalent to the Axiom of Choice.

Axiom 1.4 (Axiom of Choice: classic version)**.** *If S is a collection of pairwise disjoint nonempty sets, then there is another set T which contains exactly one element from each of the elements of S.*

Here is my favorite version.

Axiom 1.5 (Axiom of Choice: function version)**.** *If $f : X \to Y$ is an onto function, then there exists a function $g : Y \to X$ such that $f \circ g = \mathbf{1}_Y$.*

Other common versions are Zorn's lemma and the Maximum Principle. If you have never gone through and carefully shown that all of these versions are logically equivalent, it is an excellent exercise. Finally, you should be quite clear on the distinction between countable and uncountable and know the difference between ω, $\aleph_0$, and c.

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Definition 1.6 (Cardinals)**.** Two sets are said to have the same *cardinality* if there is a bijection between them, and A is larger than B if there is an injection from B to A. Then Cantor-Schoeder-Bernstein theorem shows that this definition of size is indeed an ordering (i.e. $A \leq B$ and $B \leq A$ implies A and B have the same cardinality). Assuming the axiom of choice cardinality is a *total order* (i.e. for all A and B, either $A \leq B$ or $B \leq A$). The notations $\aleph_0$ and c are used for the cardinalities of $\mathbb{N}$ (or $\mathbb{Z}$ or $\mathbb{Q}$) and $\mathbb{R}$ (or $\mathbb{C}$), respectively. Sets with cardinality $\aleph_0$ are called *countable*. Larger sets are *uncountable*. As you can see, then algebraic constructions do not change the cardinalities, but metric completion does.

You should also be familar with Cantor's diagonal argument (although we won't review it here).

Definition 1.7 (Ordinals)**.** The ordinals are well-ordered sets. This is a finer distinction among ordered sets in the sense that distinct well-ordered sets can have the same cardinality. The natural numbers with their usual ordering are the ordinal denoted by ω. Ordinal addition is unusual in the sense that it is non-commutative. $1 + \omega = \omega$ but $\omega + 1$ is different from (and bigger than) ω, even though both have the same cardinality (i.e. $\aleph_0$). Ordinal arithmetic will not be needed, but I will occasionally refer to ω.

2. Real numbers

As highlighted in the text, the key property which distinguishes the reals from the rationals is the LUB property (rewritten here as the GLB property to emphasize its similarity to (and difference from) the Well-Ordering Axiom and the Well-Ordering Principle.

Axiom 2.1 (GLB property)**.** *Every nonempty subset of the reals which is bounded below has a greatest lower bound.*

The big difference here is that while least elements are greatest lower bounds, greatest lower bounds need not be least elements for the sole reason that they may belong to the subset for which they are the glb. This axiom quickly implies the completeness of the reals as well as the Archimedian Ordering Priniciple.

Theorem 2.2 (Archimedian ordering principle)**.** *There is no real number which is an upper bound for the natural numbers.*

As we shall see later, there are other distance functions on the natural numbers which fail to have this property. These are derived from p-adic absolute values.

3. Functions

As an early motivation for the transition from real analysis to metric spaces, I briefly reviewed some of the major ideas behind Fourier analysis and tried to encourage you to think of functions as "vectors" in the following sense.

The collection $\{f : \{1\} \to \mathbb{R}\}$ is clearly in one-to-one correspondence with $\mathbb{R}$ itself. The collection $\{f : \{1, 2\} \to \mathbb{R}\}$ is in one-to-one correspondence with $\mathbb{R}^2$, where the image of 1 corresponds to the first coordinate and the image of 2 corresponds to the second coordinate. Similarly, The collection $\{f : \{1, 2, \ldots, n\} \to \mathbb{R}\}$ can be thought of as $\mathbb{R}^n$.

Remark 3.1 (Notation)**.** Combinatorialists use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ and I will use this notation where convenient, even though the text does not. Another general notation, if A and B are sets, we use A^B to mean the collection of functions from B to A. This notation matches our expectations about size. The size of A^B is the $|A|^{|B|}$.

Notice that we can use this bijection to define a natural vector space structure on $\mathbb{R}^{[n]}$ which matches that of $\mathbb{R}^n$. In particular, if f and g are two functions in $\mathbb{R}^{[n]}$ then the image of i under $f + g$ is $f(i) + g(i)$ and the image of i under αf is $\alpha \cdot f(i)$. Using this definition, the collection $\{f : X \to \mathbb{R}\}$ is a real vector space for any set X. In particular, the collection of real sequences (i.e. $\{s : \mathbb{N} \to \mathbb{R}\}$ which the book denotes (s_n)) and real-valued functions (the collection $\{f : \mathbb{R} \to \mathbb{R}\}$) are both vector spaces in a natural way.

If we try to extend the standard inner product structure on $\mathbb{R}^n$ to $\mathbb{R}^\omega$ or $\mathbb{R}^\mathbb{R}$ we get an "angle" measure which is not always defined. Cutting $\mathbb{R}^\mathbb{N}$ down to the subspace where the sum $\sum_\mathbb{N} a_n b_n$ is always finite leads to the notion of a Hilbert space. The natural analog for real valued functions is something like $\int_{-\infty}^{\infty} f(x)g(x)dx$. I'll say more about this analogy as the course continues.