NOTES FOR OCTOBER 4, 2004

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One must be able to say at all times - instead of points, straight lines and planes - tables, chairs, and beer mugs.
David Hilbert (1862-1930)

1. Equivalent metrics

We have now defined metrics d_p on $\mathbb{R}^n$ for all $p \in [1, \infty]$. How different are these metrics? Can we find a function f between $\mathbb{R}^n$ and some metric space (A, d) such that f is continuous when $\mathbb{R}^n$ is given one metric d_p but not continuous when assigned another metric d_q? The answer is no. First note that the composition of continous maps is continous (easy exercise). Next, consider the identity map from $\mathbb{R}^n$ to itself, with the domain and range given different metrics. These maps are also continuous.

Proposition 1.1. *The identity map on $\mathbb{R}^n$ is (d_p, d_q) continuous, $\forall\ p, q \in [1, \infty]$.*

Proof. Every d_p (or d_q) ball contains a diamond and is contained in a square. As a result, every d_p-ball of radius ϵ contains a d_q-ball (centered at the same point) of radius δ for some $\delta > 0$. □

Corollary 1.2. *If (A, d) is a metric space and $f : A \to \mathbb{R}^n$ is a function, then f is (d, d_p) continuous if and only if f is (d, d_q) continuous for all $p, q \in [1, \infty]$. Similarly, if (C, d') is a metric space and $g : \mathbb{R}^n \to C$ is a function, then g is (d_p, d') continuous if and only if g is (d_q, d') continuous for all $p, q \in [1, \infty]$.*

Because of this, the metrics d_p on $\mathbb{R}^n$ for all $p \in [1, \infty]$ are considered equivalent.

Remark 1.3. The same holds for the L^p metrics on $\mathcal{C}([a, b], \mathbb{R})$, $p \in [1, \infty]$.

Definition 1.4 (Equivalent metrics)**.** Two metrics d_1 and d_2 on B are *equivalent* if and only for all functions $f : A \to B$ and $g : B \to C$ and for all metrics d and d' on A and C, f is (d, d_1) continuous if and only if f is (d, d_2) continuous and g is (d_1, d') continuous if and only if g is (d_2, d') continous.

Lemma 1.5. *d_1 and d_2 are equivalent metrics on B if and only if the identity function on B is (d_1, d_2) continuous and (d_2, d_1) continous, which is true if and only if every d_1-ball contains some d_2-ball centered at the same point and every d_2-ball contains some d_1-ball centered at the same point.*

Definition 1.6 (Homeomorphism)**.** A bijection $f : A \to B$ (between metric spaces) is called a *topological equivalence* or *homeomorphism* if both f and f^{-1} are continuous.

The previous lemma, restated says that two metrics on B are equivalent if and only if the identity map is a homeomorphism.

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2. Open sets

Lemma 2.1. *Let (A, d) be a metric space. For every $y \in B_\epsilon(x)$ there is a δ such that $B_\delta(y) \subset B_\epsilon(x)$.*

Definition 2.2 (Open sets)**.** A set U is open if for every point $x \in U$ there is a open ball containing x and contained in U.

By the previous result, open balls are open sets. Also notice that if d and d' are equivalent metrics then balls in the d metric are open sets in the d' metric and vice versa.

Previously defined properties can be redefined in terms of open sets.

Proposition 2.3. *A map $f : (A, d) \to (B, d')$ is continuous if and only if the inverse image of each open set is open.*

Proposition 2.4. *Two metrics are equivalent if and only if they define the same collection of open sets.*

What properties do the open sets satisfy? It isn't too hard to see that the collection of open sets in a metric space are closed under fininte intersections and arbitrary unions. Trivially, the empty set and the whole space are open sets. This is now our definition of a topology.