NOTES FOR OCTOBER 1, 2004

JON MCCAMMOND

Those properties which, in the theory of ordinary numerical functions, arise only in the handling of the most complicated, most difficult, least practical functions, are the characteristics that are the simplest, most practical, in fact indispensable attributes of functionals or functions of generalized variables.

Maurice Fréchet (1878-1973)

In his 1906 Dissertation, Maurice Fréchet introduced the notion of a metric space, as well as the notions of compactness, uniformly continuity, equi-continuity and metric space completion. It was quite a good dissertation. His primary motivation was to generalize the notion of convergence so that he could extend theorems like the Extreme Value Theorem to collections of functions so that they could be used in the calculus of variations.

1. Metrics and metric spaces

Most of the early examples of metric spaces satisfy the triangle inequality because of a single, fairly general, inequality due to Minkowski. Here is one version taken from (green) Rudin, p.63.

Theorem 1.1 (Minkowski's inequality)**.** *If p is in $(1,\infty)$, X is a measure space with measure μ, and f and g are measurable functions on X with range $[0,\infty]$ then*

$$\left\{\int_X (f+g)^p \ d\mu\right\}^{1/p} \leq \left\{\int_X f^p \ d\mu\right\}^{1/p} + \left\{\int_X g^p \ d\mu\right\}^{1/p}$$

As a corollary, the function

$$d(f,g) = \left\{\int_X |f-g|^p \ d\mu\right\}^{1/p}$$

is a pseudo-metric on the set of measurable functions on X with

$$\left\{\int_X |f|^p \ d\mu\right\}^{1/p} < \infty$$

This one example includes the ℓ^p metrics on $\mathbb{R}^n$, the ℓ^p metrics on various subspaces of the space of sequences, and the L^p metrics on the various subspaces of the space of real-valued functions.

Date: October 1, 2004.