

Name:

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Math 2E: Quiz 2 Solutions

- (2) 1. Let f and g be two twice differentiable scalar fields. Compute $\operatorname{div}(\nabla f \times \nabla g)$.

Proof: $\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g)$. We know that ∇f , and ∇g are both curl free, hence $\operatorname{curl}(\nabla f) = \operatorname{curl}(\nabla g) = 0$. Therefore $\operatorname{div}(\nabla f \times \nabla g) = 0$ \square .

- (4) 2. Let $\mathbf{F} = \langle xy \sin(z), \cos(xz), y \cos(z) \rangle$, S be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Calculate the flux of \mathbf{F} across S .

Solution: The flux of \mathbf{F} across S is given by the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$. S is smooth surface with outward normal and the vector field \mathbf{F} is differentiable on \mathbb{R}^3 , so the divergence theorem applies. Now

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(xy \sin(z)) + \frac{\partial}{\partial y}(\cos(xz)) + \frac{\partial}{\partial z}(y \cos(z)) = y \sin(z) + 0 - y \sin(z) = 0$$

Let E be the region that is enclosed by the ellipsoid.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F}) \, dV = \iiint_E 0 \, dV = 0 \quad \square$$

- (4) 3. Let S be the hemisphere $x = \sqrt{1 - y^2 - z^2}$ oriented in the direction of the positive x-axis and $\mathbf{F} = \langle e^{xy} \cos(z), x^2 z, xy \rangle$. Evaluate the following:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

Solution: First notice that S is a smooth surface whose boundary is described by $y^2 + z^2 = 1$, which itself is a smooth curve. The vector field \mathbf{F} is differentiable on all of \mathbb{R}^3 , hence Stokes' theorem applies. Parameterizing the boundary keeping positive orientation we have $\mathbf{r}(t) = \langle 0, \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$. So we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{\partial S} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} \langle e^0 \cos(\sin(t)), 0, 0 \rangle \cdot \langle 0, -\sin(t), \cos(t) \rangle \, dt \\ &= \int_0^{2\pi} 0 \, dt = 0 \end{aligned}$$