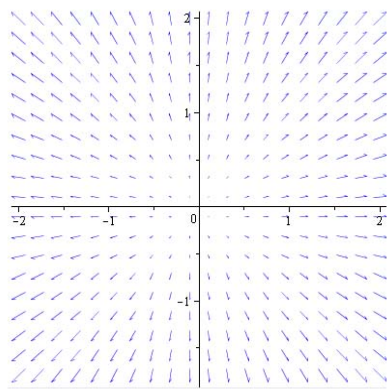
Math 2E
Final Solutions: by Jim

1) Find and sketch the gradient vector field for $f(x, y) = x^2 + y^2$, make sure the directions and magnitudes of the vectors are correct.

Solution The gradient is given by $\nabla(f) = \langle 2x, 2y\rangle$. Which is 2 times the position vector $\langle x, y\rangle$. So the magnitude increases as the distance from $(0, 0)$ increases, which can be seen below.



2) Show that the vector field $F(x, y) = \langle e^{-y}, -xe^{-y}\rangle$ is conservative and find a scalar potential function f for $F(x, y)$. Then Evaluate the line integral $\displaystyle\int_C F \cdot dr$, where C is the part of the parabola $y = x^2$ connecting $(0, 0)$ to $(1, 1)$.

Solution First notice that the domain of $F(x, y) = \langle P(x, y), Q(x, y)\rangle$ is all of $\mathbb{R}^2$ which is an open simply-connected region. So $F(x, y)$ is conservative if $\nabla \times F = (Q_y - P_x)\mathbf{k} = 0$.

$$Q_x = -e^{-y} = P_y$$

Hence F is conservative. For its potential function

$$f_x = e^{-y} \quad \Rightarrow \quad f = xe^{-y} + g(y)$$

differentiating with respect to y we have

$$f_y = -xe^{-y} + g'(y) \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad f = xe^{-y} + K$$

For the line integral, a parameterzation for the curve is $r(x) = \langle x, x^2\rangle$ for $0 \leq x \leq 1$. Now $r(0) = (0, 0)$, $r(1) = (1, 1)$. Now since F is a conservative vector field the fundemental theorem for line integrals applies and the line integral becomes:

$$\int_C F \cdot dr = \int_C \nabla(f) \cdot dr = f(r(b)) - f(r(a)) = f(1, 1) - f(0, 0) = e^{-1}$$

3) Show that the vector field $F(x, y, z) = \langle \frac{y}{1+x^2} + \tan^{-1}(z), \tan^{-1}(x), \frac{x}{1+z^2}\rangle$ is conservative, and then evaluate the line integral $\displaystyle\int_C F \cdot dr$, where C is the intersection of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ and the cylinder $x^2 + y^2 = 1$, in counter-clockwise direction.

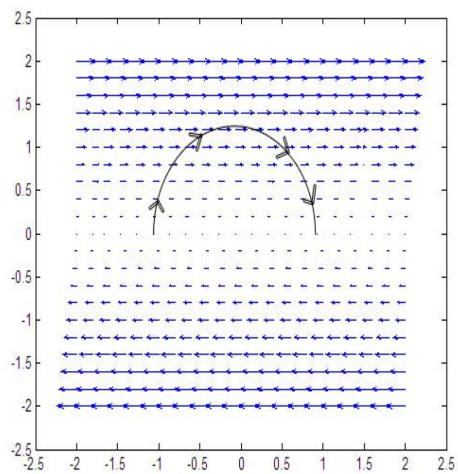
Solution First notice that the domain for $F = \langle P, Q, R\rangle$ is all of $\mathbb{R}^3$, which is an open simply connected region. So F is conservative if $\nabla \times F = 0$.

$$\begin{aligned}
\nabla \times F &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{1+x^2} + \tan^{-1}(z) & \tan^{-1}(x) & \dfrac{x}{1+z^2} \end{vmatrix} \\
&= \langle 0, \frac{1}{1+z^2} - \frac{1}{1+z^2}, \frac{1}{1+x^2} - \frac{1}{1+x^2}\rangle = \langle 0, 0, 0\rangle
\end{aligned}$$

So F is conservative. For a parameterzation for the curve of intersection notice that when you subtract the two surfaces you get $z^2 = 3$, which implies that $z = \sqrt{3}$, since $z \geq 0$. So $r(t) = \langle \cos(t), \sin(t), \sqrt{3} \rangle$. Now for any starting point a and ending point b on the curve C we have $r(a) = r(b)$. Hence our line integral is

$$\int_C F \cdot dr = \int_C \nabla(f) \cdot dr = f(r(b)) - f(r(a)) = 0$$

4) The vector field $F(x, y)$ is given in the figure, determine if $\int_C F \cdot dr$ is positive, negative or zero, where C is the upper half circle connecting $(-1, 0)$ to (1,0). Give reasons.



Solution C has positive orientation and flows in the same direction as F.

$$\int_C F \cdot dr = \int_C F \cdot T ds > 0$$

5) Find the curl and divergence of the vector field $F = \langle z^2 e^{-x}, y^3 \ln(z), x e^{-y} \rangle$.

Solution

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 e^{-x} & y^3 \ln(z) & x e^{-y} \end{vmatrix} = \langle -x e^{-y} - \frac{y^3}{z}, 2z e^{-x} - e^{-y}, 0 \rangle$$

$$\nabla \cdot F = -z^2 e^{-x} + 3y^2 \ln(z)$$

6) Find an explicit equation $z = f(x, y)$ and given restrictions when necessary for the surface defined by the parametric equation $r(u, v) = (u, v^2, u^3)$.

Solution From the vector equation $r(u, v) = (u, v^2, u^3)$, we have the paramentric equations

$$x = u, y = v^2, z = u^3 \quad \Rightarrow \quad z = f(x, y) = x^3, y \geq 0$$

7) The surface S is the portion of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 9$, find a parametric equation for S and find the surface area of S.

Solution Subtracting the two equations we have $z^2 = 16$, which implies that $z = \pm 4$. Switching both equations to spherical coordinates we have

$$r(\phi, \theta) = \langle 5 \sin(\phi) \cos(\theta), 5 \sin(\phi) \sin(\theta), 5 \cos(\phi) \rangle$$

and since $z = 5\cos(\phi) = \pm 4$, we have $\phi = \cos^{-1}(\pm\frac{4}{5})$. So the parameterzation is given above with

$$0 \leq \theta \leq 2\pi, \quad \phi \in [0, \cos^{-1}(\frac{4}{5})] \cup [\cos^{-1}(-\frac{4}{5}), \pi]$$

Now by symmetry, the surface area is given by

$$\iint_S dS = \iint_D \|r_\phi \times r_\theta\| dA = 2\int_0^{2\pi}\int_0^{\cos^{-1}(\frac{4}{5})} 25\sin(\phi)d\phi d\theta$$

Using Fubini's theorem we have

$$50\int_0^{2\pi} d\theta \int_0^{\cos^{-1}(\frac{4}{5})} \sin(\phi)d\phi = -100\pi\cos(\theta)\Big|_0^{\cos^{-1}(\frac{4}{5})} = -80\pi + 100\pi = 20\pi$$

8) Use a change of variables to evaluate the following integrals, given the transformations: $x = f(u,v), y = g(u,v)$.

a) $\displaystyle\iint_D y^3(2x-y)\cos(2x-y)dA$, where D is the region bounded by the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,2)$ and $(1,2)$. $f(u,v) = \frac{1}{2}(u+v)$, $g(u,v) = v$.

Solution If $x = \frac{1}{2}(u+v)$, $y = v$, then $u = 2x - y$, $y = v$, so the region bounded by the parallelogram can be described in the uv-plane as $D_{uv} = \{(u,v) : 0 \leq u \leq 4, 0 \leq v \leq 2\}$. Now

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{array}\right| = \frac{1}{2}$$

So the integral becomes

$$\begin{aligned} \iint_D y^3(2x-y)\cos(2x-y)dA &= \int_0^2\int_0^4 v^3 u\cos(u)\frac{1}{2}dudv \\ &= \frac{1}{2}\int_0^2 v^3 dv\int_0^4 u\cos(u)du = 2\left(\cos(4) + 4\sin(4) - 1\right) \end{aligned}$$

b) $\displaystyle\iint_D e^{\frac{x+y}{x-y}}dA$, where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$. $f(u,v) = \frac{1}{2}(u+v)$, $g(u,v) = \frac{1}{2}(u-v)$.

Solution If $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$, then $u = x+y$, $v = x-y$ so the trapezoidal region can be described in the uv-plane as $D_{uv} = \{(u,v) : 1 \leq v \leq 2, -v \leq u \leq v\}$. Now

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

So the integral becomes

$$\begin{aligned} \iint_D e^{\frac{x+y}{x-y}}dA &= \int_1^2\int_{-v}^{v} e^{\frac{u}{v}}\frac{1}{2}dudv \\ &= \frac{1}{2}\int_1^2 (e - e^{-1})vdv = \frac{3}{4}\left(e - e^{-1}\right) \end{aligned}$$

9) Evaluate $\displaystyle\int_C x^2 + y^2 ds$ where C is given by $C : x = e^{-t}\cos(t)$, $y = e^{-t}\sin(t)$, $0 \leq t \leq \dfrac{\pi}{2}$

Solution Let $f(x,y) = x^2 + y^2$, the vector equation for C is $r(t) = \langle e^{-t}\cos(t), e^{-t}\sin(t)\rangle$. So $f(r(t)) = e^{-2t}$ and $r'(t)$ is given by

$$r'(t) = \langle -e^{-t}(\cos(t) + \sin(t)), e^{-t}(\cos(t) - \sin(t))\rangle, \quad \|r'(t)\| = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} = \sqrt{2}e^{-t}$$

Now the line integral

$$\int_C f(x,y)ds = \int_0^{\pi/2} f(r(t))\|r'(t)\|dt = \sqrt{2}\int_0^{\pi/2} e^{-2t}e^{-t}dt = \frac{\sqrt{2}}{3}\left(1 - e^{\frac{-3}{2}\pi}\right)$$

10) Let $F = \langle y, -x\rangle$ and let C_1, C_2 be the following two paths joining $(0,0)$ to $(1,1)$. $C_1 : y = x$, $C_2 : y = x^2$. Show that $\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$. Explain what this means.

Solution A parameterzation for C_1 is $r_1(x) = \langle x, x\rangle$, and so

$$\int_{C_1} F \cdot dr = \int_0^1 \langle x, -x\rangle \cdot \langle 1, 1\rangle dx = \int_0^1 0dx = 0$$

A parameterzation for C_2 is $r_2(x) = \langle x, x^2\rangle$, and so

$$\int_{C_2} F \cdot dr = \int_0^1 \langle x^2, -x\rangle \cdot \langle 1, 2x\rangle dx = -\int_0^1 x^2 dx = -\frac{1}{3}$$

First it means that the vector field is not conservative. Second the work done by the field along C_1 is 0, while work done by the field along C_2 is negative.

11) Use Stokes Theorem to evaluate $\int F \cdot dr$, where $F = \langle x - z, y - x, z - y\rangle$ and C is the boundary of the triangular region with vertices $(12, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 12)$ traversed counterclockwise as viewed from above the origin.

Solution Curl$(F) = \langle -1, -1, -1\rangle$. Two vectors that span the plane of the triangular region are $\langle 12, -3, 0\rangle$, and $\langle 12, 0, -12\rangle$, so the normal of this plane is the cross product of these two vectors which is $n = \langle 1, 4, 1\rangle$. Using the point$(12, 0, 0)$, we have the equation of the plane as $x + 4y + z = 12$. Now the triangular region S is enclosed by $D = \{(x, y) : 0 \leq x \leq 12, 0 \leq y \leq 3 - \frac{x}{4}\}$. Let $z = g(x, y) = 12 - x - 4y$, and let Curl$(F) = \langle P, Q, R\rangle = \langle -1, -1, -1\rangle$ we have

$$\begin{aligned}
\int_C F \cdot dr = \iint_S \nabla \times F d\bar{S} &= \iint_D -Pg_x - Qg_y + RdA \\
&= \int_0^{12} \int_0^{3-\frac{x}{4}} -(-1)(-1) - (-1)(-4) + (-1)dydx \\
&= \int_0^{12} \int_0^{3-\frac{x}{4}} -6dydx \\
&= 3\int_0^{12} \frac{x}{2} - 6dx \\
&= 3\left(\frac{x^2}{4} - 6x\right)_0^{12} = -108
\end{aligned}$$

12) Use Divergence Theorem to find the flux of the field $F = \langle \cos(yz), e^{xz}, 3z^2\rangle$ across the surface S given by the hemisphere, $x^2 + y^2 + z^2 = 4$, $z \geq 0$ together with the disk $x^2 + y^2 = 4$ in the xy-plane.

Solution The flux of the field F across S is given by

$$\iint_S F \cdot d\bar{S} = \iiint_E \nabla \cdot F dV$$

by the Divergence Theorem. So $\nabla \cdot F = 6z$. Parameterizing our region E in spherical coodinates we have $E = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi\}$ Hence our integrals are

$$\begin{aligned}
\iint_S F \cdot d\bar{S} = \iiint_E 6zdV &= 6\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho\cos(\phi)\rho^2 \sin(\phi)d\rho d\phi d\theta \\
&= 6\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin(\phi)\cos(\phi) \int_0^2 \rho^3 d\rho \\
&= 6(2\pi)(\frac{1}{2})(4) = 24\pi
\end{aligned}$$