Name:
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Math 147: Quiz 7

(5) **1.** Let f be a non-constant entire function. Prove that if $\lim_{|z|\to\infty} |f(z)| = \infty$, then $|f|$ must be a polynomial.

Solution: Consider $g(z) = f\left(\frac{1}{z}\right)$, then $\lim_{z\to 0} g(z) = \infty$. Now suppose that $g(z)$ has a pole of order k and consider the Laurent expansion:

$$g(z) = \frac{a_{-k}}{z^k} + \frac{a_{-k+1}}{z^{k-1}} + \cdots + a_0 + a_1 z + \cdots \quad \Rightarrow \quad z^k g(z) = z^k \sum_{n=-k}^{\infty} a_n z^n$$

Now $|z^k g(z)| \to c = f(0)$ as $|z| \to \infty$. This implies by continuity that $|z^k g(z)| \leq (|c|+1) z^k$ for large z. Hence $z^k g(z)$ is a polynomial of at most degree k. Now we have:

$$g(z) z^k = \sum_{n=0}^{k} a_n z^n \quad \Rightarrow \quad f\left(\frac{1}{z}\right) = g(z) = \sum_{n=0}^{k} \frac{a_n}{z^{k-n}}$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{k} a_n z^{k-n}$$

(5) **2.** Show that for $R > 0$, there is N_R such that when $n > N_R$, the function

$$P_n(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} \neq 0, \quad \forall\ |z| \leq R.$$

Solution: First notice that $P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ and that $P_n(z) \to e^z$ uniformly as $n \to \infty$ on compact sets of $\mathbb{C}$. Fix $R > 0$

$$\forall \epsilon > 0\ \exists N_R\ s.t.\ \left| \sum_{k=0}^{n} \frac{z^k}{k!} - \sum_{k=0}^{m} \frac{z^k}{k!} \right| = \left| \sum_{k=m}^{n} \frac{z^k}{k!} \right| \leq \epsilon, \quad \forall n > m > N_R.$$

This implies that

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| < \epsilon, \quad \forall n > N_R$$

which implies that

$$1 \leq |e^z| < \epsilon + \left| \sum_{k=0}^{n} \frac{z^k}{k!} \right|, \quad \forall n > N_R, \forall z \in \overline{D(0,R)}$$

$$\therefore \forall R > 0\ \exists N_R\ s.t.\ \sum_{k=0}^{n} \frac{z^k}{k!} \neq 0, \quad \forall n > N_R\ \square$$