Complex analysis, Fun problems:

Exercise 1. Prove or disprove that there is a sequence of analytic polynomials $\{p_n(z)\}, n \in \mathbb{N}$, so that $p_n(z) \to \bar{z}^4$ as $n \to \infty$ uniformly for $z \in \partial D(0,1)$.

Solution: The statement is not true. Suppose that there exists such a sequence of analytic polynomials such that $p_n(z) \to \bar{z}^4$. Then for all n we have $\frac{d}{d\bar{z}}p_n(z) = 0$ since $p_n(z)$ is analytic. However $\frac{d}{d\bar{z}}\bar{z}^4 = 4\bar{z}^3 \neq 0$ for all $z \in \mathbb{C}$. Clearly $0 \not\to 4\bar{z}^3$ for all $z \in \mathbb{C}$ $\square$

Exercise 2. Let $U \subset \mathbb{C}$ be a connected open set, and γ be a closed curve in U. Suppose that for any function $f(z)$ holomorphic on U we have

$$\oint f(z)dz = 0.$$

Does it imply that γ is homotopic to a constant curve?

Exercise 2b. Find an example of a holomorphic function $f(z)$ on a domain U, $\gamma = \partial U$ such that

$$\oint_\gamma f(z)dz = 0.$$

Solution: No γ does not have to a constant curve, consider the function $f(z) = z^{-1}$, on the punctured disk $U = D(2,1) - \{2\}$. Then $f(z)$ is holomorphic on U, now fix $r \in (0,1)$ and let $\gamma = re^{\imath t} + 2$ for $t \in [0, 2\pi]$, then by Cauchy's theorem we have

$$\int_\gamma f(z)dz = 0.$$

but $re^{\imath t} + 2$ is clearly not a constant curve. $\square$

Exercise 3. Let $\frac{1 - e^{2\imath z}}{2z^2}$, and consider the following contour Γ:

$$\Gamma = \begin{cases} \gamma_1 := \quad t & t \in [-R, -1/R] \\ \gamma_2 := e^{\imath t}/R & t \in [\pi, 2\pi] \\ \gamma_3 := \quad t & t \in [1/R, R] \\ \gamma_4 := Re^{\imath t} & t \in [0, \pi] \end{cases}$$

Compute $\int_\Gamma f(z)\ dz$

Solution:

$$\int_\Gamma f(z)\ dz = 2\pi\imath\,\mathrm{Res}(f(z)) = 2\pi\imath \lim_{z\to 0}\frac{d}{dz}z^2 f(z) = 2\pi\imath \lim_{z\to 0} -\imath e^{2\imath z} = 2\pi\imath(-\imath) = 2\pi$$

Exercise 4 Evaluate the integral:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx$$

Hint: use problem (3)

Solution: The function has a removable singularity at $x = 0$, so consider the following

$$f(x) = \frac{1 - e^{2\imath x}}{2x^2} \qquad \Rightarrow \qquad \Re(f(x)) = \frac{1 - \cos(2x)}{2x^2} = \frac{\sin^2(x)}{x^2}$$

Now for the integral around Γ we know it's equal to 2π from part (3).

$$\int_{\gamma_1} f(z)\ dz = \frac{1}{2}\int_{-R}^{-1/R}\frac{1 - e^{2\imath t}}{t^2}\ dt \qquad \Rightarrow \qquad \frac{1}{2}\int_{-\infty}^{0}\frac{1 - e^{2\imath t}}{t^2}\ dt \text{ as } R \to \infty$$

For the integral on γ_2 we have

$$\begin{aligned}\int_{\gamma_2} f(z)\ dz &= \frac{1}{2}\int_{\pi}^{2\pi} \frac{\left(1-e^{2\imath e^{\imath t}/R}\right)\imath e^{\imath t}/R}{e^{2\imath t}/R^2} \\ &= \frac{\imath}{2}\int_{\pi}^{2\pi} \frac{1-e^{2\imath e^{\imath t}/R}}{e^{2\imath t}/R}\end{aligned}$$

Now letting $R \to \infty$ and using L'Hospitals rule we have

$$\frac{\imath}{2}\int_{\pi}^{2\pi} \frac{-e^{2\imath e^{\imath t}/R}\,(2\imath e^{\imath t}/R)\,(\imath)}{\imath e^{\imath t}/R} = \int_{\pi}^{2\pi}\ dt = \pi$$

For the integral on γ_3 we have

$$\int_{\gamma_3} f(z)\ dz = \frac{1}{2}\int_{1/R}^{R} \frac{1-e^{2\imath t}}{t^2}\ dt \quad \Rightarrow \quad \frac{1}{2}\int_{0}^{\infty} \frac{1-e^{2\imath t}}{t^2}\ dt \text{ as } R \to \infty$$

Now for γ_4 we have

$$\begin{aligned}\int_{\gamma_4} f(z)\ dz &= \frac{1}{2}\int_{0}^{\pi} \frac{1-Re^{2\imath t}}{R^2 e^{2\imath t}} R\imath e^{\imath t}\ dt \\ &= \frac{\imath}{2}\int_{0}^{\pi} \frac{1-Re^{2\imath t}}{Re^{\imath t}}\end{aligned}$$

Putting this all together we have

$$2\pi = \pi + \frac{1}{2}\int_{-\infty}^{0} \frac{1-e^{2\imath t}}{t^2}\ dt + \frac{1}{2}\int_{0}^{\infty} \frac{1-e^{2\imath t}}{t^2}\ dt$$

Taking real parts we have

$$\pi = \int_{-\infty}^{0} \frac{\sin^2(t)}{t^2}\ dt + \int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt = 2\int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt$$

Hence we have $\displaystyle\int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt = \frac{\pi}{2}$ $\square$

Exercise 5 Let $f(z)$ be analytic on $\mathbb{C} - \{1\}$ and have a simple pole at $z = 1$ with residue λ. Prove that for every $R > 0$,

$$\lim_{n\to\infty} R^n \left|(-1)^n \frac{f^{(n)}(2)}{n!} - \lambda\right| = 0$$

Hint: Look at Laurent expansion, and define a suitable entire auxiliary function. What can you say about the Taylor series of this function?

Proof: Since $f(z)$ has simple pole at one we have the Laurent expansion.

$$f(z) = \frac{\lambda}{z-1} + \sum_{n=0}^{\infty} a_n (z-1)^n$$

Define $g(z) = f(z) - \dfrac{\lambda}{z-1}$, then $g(z)$ is an entire function. Now for $|z-2| < 1$ f(z) has the taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(z-2)^2$$

Also we have the geometric series for $\frac{\lambda}{z-1}$

$$\frac{\lambda}{z-1} = \frac{\lambda}{1+(z-2))} = \lambda\sum_{n=0}^{\infty}(-1)^n(z-2)^n$$

This implies that the series for $g(z)$ about 2 is

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(z-2)^n - \lambda \sum_{n=0}^{\infty}(-1)^n(z-2)^n = \sum_{n=0}^{\infty}(z-2)^n \left(\frac{f^{(n)}(2)}{n!} - \lambda(-1)^n\right)$$

Now for $|z-2| < 1$ we have that $(z-2)^n \left|\frac{f^{(n)}(2)}{n!} - \lambda(-1)^n\right| \to 0$. But since $g(z)$ is entire we this holds for any $z \in \mathbb{C}$ Hence for any $R > 0$ we have

$$R^n \left|\frac{f^{(n)}(2)}{n!} - \lambda(-1)^n\right| \to 0 \text{ as } n \to \infty$$

So the result is shown. $\square$

Exercise 6 Prove the Lioville's theorem by calculating the following integral

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz$$

and taking the limit $R \to \infty$.

Solution: Suppose that $f(z)$ is a bounded entire function, that is, there exists $M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$. Fix $R > 0$, now for $a, b \in D(0, R)$ the integral is bounded by;

$$\left|\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz\right| \leq \frac{2\pi RM}{(R-|a|)(R-|b|)} \to 0 \text{ as } R \to \infty$$

Now by direct computation we have

$$\begin{aligned} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz &= 2\pi\imath \left(\operatorname{Res} f(a) + \operatorname{Res} f(b)\right) \\ &= 2\pi\imath \frac{f(b)-f(a)}{b-a} = 0 \end{aligned}$$

This implies that $f(b) = f(a)$ for all $a, b \in \mathbb{C}$, hence $f(z)$ is constant.

Exercise 7 Show that there is a holomorphic function defined in the set

$$\Omega = \{z \in \mathbb{C} : |z| > 4\}$$

Whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Hint: Residue theorem

Solution: Let γ be a closed curve lying outside of Ω. Now if there exists such a function $F(z)$, such that

$$F'(z) = f(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

then the following condition should be satisfied

$$\int_\gamma F'(z) = 0$$

It suffices to show this is true for $\gamma = re^{\imath t}$, where $r > 0$. Now $F'(z)$ has 3 simple poles $\{1, 2, 3\}$ lying inside of γ. Hence we have

$$\begin{aligned} \int_\gamma f(z) &= 2\pi\imath \left(\operatorname{Res} f(1) + \operatorname{Res} f(2) + \operatorname{Res} f(3)\right) \\ &= 2\pi\imath \left(\frac{1}{2} - 2 + \frac{3}{2}\right) = 0 \end{aligned}$$

Hence by Morera's there does exists such a function $F(z)$ such that $F'(z) = f(z)$ $\square$

Exercise 8 Prove or disprove each of the statements:

(a) If f is a function on the unit disk D such that $f^2(z)$ is analytic on D, then f itself is analytic.

Solution: This statement is false. Let $f(z) = \sqrt{z}$, then $f^2(z) = z$ which is holomorphic on $D(0,1)$, but $f(z)$ is not holomorphic at $z = 0$.

(b) If $f(z)$ is a continuously differentiable function on D, and if $f^2(z)$ is analytic on D, then $f(z)$ itself is analytic.

Proof: Let $f(z) = f(x,y) = u(x,y) + \imath v(x,y)$ where u, v are harmonic. Then $f^2(z) = u^2(z) - v^2(z) + 2\imath u(z)v(z)$. Define g(z) and h(z) as follows:

$$g(z) := \Re(f^2(z)) = u^2(z) - v^2(z)$$
$$h(z) := \Im(f^2(z)) = 2u(z)v(z)$$

Now since $f^2(z)$ is holomorphic we know that $f^2(z)$ satisfies the Cauchy-Riemann equations, i.e.,

$$\frac{dg}{dx} = \frac{dh}{dy} \qquad \frac{dg}{dy} = -\frac{dh}{dx}$$

Computing the above we have

$$\frac{dg}{dx} = 2u\frac{du}{dx} - 2v\frac{dv}{dx} = \frac{dh}{dy} = 2u\frac{dv}{dy} + 2v\frac{du}{dy}$$

this implies that

$$u\left(\frac{du}{dx} - \frac{dv}{dy}\right) = v\left(\frac{du}{dy} + \frac{dv}{dx}\right) \tag{0.1}$$

Computing the other equality we have

$$\frac{dg}{dy} = 2u\frac{du}{dy} - 2v\frac{dv}{dy} = -\frac{dh}{dx} = -2u\frac{dv}{dx} - 2v\frac{du}{dx}$$

which implies that

$$u\left(\frac{du}{dy} + \frac{dv}{dx}\right) = v\left(\frac{dv}{dy} - \frac{du}{dx}\right) \tag{0.2}$$

Solving the above system of equations in (1) and (2) implies that for all $z \in D(0,1)$ either,

$$u^2 + v^2 = 0 \qquad \text{or} \qquad \frac{du}{dx} = \frac{dv}{dy}, \frac{du}{dy} = -\frac{dv}{dx}$$

The former case implies that $u = v = 0$, which implies $f(z) = 0$ and thus $f(z)$ is analytic in $D(0,1)$. The latter case implies $f(z)$ satisfies the Cauchy-Riemann equations, and thus $f(z)$ is analytic in $D(0,1)$. Either way $f(z)$ is analytic in $D(0,1)$. $\square$