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Math 147: Quiz 4

(5) **1.** Let $\{f_j\}$ be a sequence of holomorphic functions on an open set $U \subset \mathbb{C}$, suppose that $f_j \to f$ uniformly on compact subsets of U. Show that

$$\frac{\partial}{\partial z} f_j \to \frac{\partial}{\partial z} f$$

uniformly on compact subsets of U. *Hint:* $\bar{D}(p,r) \subset U$ *is a compact subset of* U.

Proof: Let $p \in U$ and let $r > 0$ such that $\bar{D}(p,r) \subset U$. It suffices to show just for this disk $\bar{D}(p,r)$. Using Cauchy's integral formula we have

$$\begin{aligned}
\lim_{j\to\infty} \frac{\partial}{\partial z} f_j(p) &= \lim_{j\to\infty} \frac{1}{2\pi\imath} \int_{\partial D(p,r)} \frac{f_j(z)}{(z-p)^2}\, dx \\
\text{by uniform continuity} &= \frac{1}{2\pi\imath} \int_{\partial D(p,r)} \lim_{j\to\infty} \frac{f_j(z)}{(z-p)^2}\, dx \\
\text{taking the limit} &= \frac{1}{2\pi\imath} \int_{\partial D(p,r)} \frac{f(z)}{(z-p)^2}\, dx \\
&= \frac{\partial}{\partial z} f(p)
\end{aligned}$$

Since p is arbitrary, we can conclude that $\frac{\partial}{\partial z} f_j \to \frac{\partial}{\partial z} f$ on compact sets of U $\square$.

(5) **2.** Let $f : D(0,2) \to \mathbb{C}$ such that $f(z)$ is analytic and $|f| \leq 3$ on $D(0,2)$. Derive the good estimate for

$$\left| \frac{\partial^2 f(\imath)}{\partial z^2} \right|$$

Solution: If we apply the Cauchy estimates we have

$$\left| \frac{\partial^2 f(\imath)}{\partial z^2} \right| \leq \frac{3 \cdot 2! \cdot r}{\min_{z \in \partial D(0,2)} (|z-i|)^3} = \frac{M \cdot 2! \cdot 2}{1^3} = 12 \ \square$$

Another approach is to use Cauchy's formula and the ML estimate (this is how of the Cauchy estimates are proved).

$$\begin{aligned}
\left| \frac{\partial^2 f(\imath)}{\partial z^2} \right| &= \left| \frac{2!}{2\pi\imath} \int_{\partial D(0,2)} \frac{f(z)}{(z-\imath)^3}\, dz \right| \\
&\leq \frac{1}{\pi} \int_{\partial D(0,2)} \left| \frac{f(z)}{(z-\imath)^3} \right| dz \\
&\leq \frac{3}{\pi} \int_{\partial D(0,2)} \frac{1}{|z-\imath|^3}\, dz \\
&= \frac{3}{\pi} \int_{\partial D(0,2)} \frac{1}{|r|^3}\, dz
\end{aligned}$$

Where $r = \min_{z \in \partial D(0,2)} |z - \imath| = 1$, so we have

$$\left| \frac{\partial^2 f(\imath)}{\partial z^2} \right| \leq \frac{6}{\pi} \int_{\partial D(0,2)} dz = 12$$