Problem: Let $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = \langle F_1(x), \ldots, F_n(x) \rangle$. Suppose that each $F_i(x) \in C^1(\mathbb{R}^n)$ and $F(0) = 0$. Finally suppose that

$$\sum_{k,j=1}^{n} \left| \frac{\partial}{\partial x_j} F_k(0) \right|^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \left| \frac{\partial}{\partial x_j} F_k(0) \right|^2 = c < 1$$

Prove that there exists an open ball B centered around the origin such that $F(B(0,\delta)) \subset B(0,\delta)$.

Solution: Fix $A \in (\sqrt{(c)}, 1)$, It suffices to show that there exists $\delta > 0$ s.t. $\|F(x)\| \leq A\|x\|$, whenever $\|x\| < \delta$. (why?)

The first order Taylor expansion for x in a neighborhood $B(0,r)$ is given by:

$$F(x) = DF(0)(x) + E(x), \text{ where } E(x) = o(\|x\|) \text{ (what is } E(x)\text{?)}$$

Now let $\{e_j\}$ be the standard basis, then we have:

$$DF(0)e_j = \begin{pmatrix} \partial_{x_j} F_1(0) \\ \vdots \\ \partial_{x_j} F_n(0) \end{pmatrix}$$

(what is the purpose of the above?)

and so

$$\|DF(0)e_j\|^2 \leq \sum_{k=1}^{n} |\partial_{x_j} F_k(0)|^2 \quad \text{and} \quad \sum_{j=1}^{n} \|DF(0)e_j\|^2 \leq \sum_{j,k=1}^{n} |\partial_{x_j} F_k(0)|^2 = c$$

(justify these inequalities)

Therefore we have $\|DF(0)x\| \leq \sqrt{c}\|x\|$, for any $x \in B(0,r)$ (why?)

Now there exists a $\delta > 0$ s.t. $\|E(x)\| < (A - \sqrt{c})\|x\|$ for all $\|x\| < \delta$(why?)

Hence we have $\|F(x)\| \leq A\|x\|$, and thus $F(B(0,\delta)) \subset B(0,A\delta) \subset B(0,\delta)$ $\square$.