

A couple of the proofs I went over in discussion. If you want to see them again come to my office hours.

1.2.12 Let $p = \frac{a+b}{2}, q = \frac{b+c}{2}, r = \frac{c+a}{2}$ compute $\frac{1}{3}(p+q+r)$ and compare with $\frac{1}{3}(a+b+c)$.

1.5.1(ii) Let R be the set described. R^c is open, hence R is closed. To show R^c is open, show that every point in R^c is contained in the interior of R^c (I did this in discussion).

For unbounded, notice that $(n, 0) \in R$ for all n , so consider the limit of $(n, 0)$ as $n \rightarrow \infty$.

For connected, use the definition of disconnected. A set S is called disconnected if $S = A \cup B$ where A and B are disjoint open sets. Suppose R is disconnected, then there exists disjoint A, B open in R such that $A \cup B = R$, now let $p \in A, q \in B$ and consider a convex combination. Look at what happens at $\partial A, \partial B$ of this combination to obtain a contradiction. (remember R is open in \mathbb{R} , but not open in $\mathbb{R}(\text{the reals})$).

For ∂R consider $\overline{R^c} \cap \overline{R}$

1.6.3 here's a proof of something similar:

<http://en.wikipedia.org/wiki/Banach%20fixed%20point%20theorem#Proof>

change $d(x, y)$ to the euclidean norm $\|x - y\|$, and the proof is the same.

1.6.16 I did this in discussion.

1.7.4 Let

$$\tilde{A} = \sup_{x \in A} (x : \|x - y\| = \text{diam}(A), y \in A) \quad \tilde{B} = \sup_{x \in B} (x : \|x - y\| = \text{diam}(B), y \in B)$$

Now let a, b be such that

$$a, b = \inf_{a \in \tilde{A}, b \in \tilde{B}} \{\|a - b\|\}$$

and let \tilde{a} be such that $\|a - \tilde{a}\| = \text{diam}(A)$.

If $\|a - b\| = 0$, then we are done. Otherwise suppose $\|a - b\| > 0$, let $y \in B(a, \|b - a\|)$. Then we have

$$\begin{aligned} \|\tilde{a} - y\| &\leq \|\tilde{a} - b\| + \|b - y\| \\ \Rightarrow \|\tilde{a} - y\| - \|\tilde{a} - b\| &\leq \|b - y\| \leq \|b - \tilde{b}\| \\ \Rightarrow \|\tilde{a} - a\| &\leq \|\tilde{a} - y\| \leq \|b - \tilde{b}\| \end{aligned}$$

Go through and justify what I did. Note, this isn't the only way to prove this statement.

1.7.9) From the problem consider the sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$, all are bounded and monotonic by definition of the set R_n . Now let $S_k = \bigcap_{n=1}^k R_n$, how are the limits of the sequences related to the limit of S_k ?

1.7.10 Let $\{x_n\}$ be a bounded sequence of real numbers with one limit point. Use the definition of a limit point, and the definition of cauchy sequence in a complete space to show convergent.

If the sequence is unbounded the statement is not true. Consider $\{n\}$ for nonnegative integers.

1.8.5 (\Rightarrow) Compact implies closed and bounded, closed implies every sequence has a convergent subsequence.

(\Leftarrow) Consider this Definition: a set $S \subset \mathbb{R}^n$ is sequentially compact if every sequence in S has a subsequence which converges to an element of S .

Claim: Sequentially compact iff closed and bounded iff compact.

Proof: Let S be sequentially compact. If S is not bounded, then there exists a sequence $\{x_n\}$ which diverges to $\pm\infty$. But this implies that every nonconstant subsequence of $\{x_n\}$ goes to infinity which contradicts definition of sequentially compact. Thus, S is bounded. Since every sequence of S converges to an element of S , S must be closed, and therefore compact (Heine-Borel).

1.8.6 Compact implies closed and bounded in \mathbb{R}^n . Proceed by contradiction, use projections back into the sets A and B .

1.8.9c part c is a special case of Tychonoff's theorem. Here is the proof for those that are interested:

<http://planetmath.org/?op=getobj&from=objects&id=7288>

For an outline. Let F be an open cover for S . S being compact there are finite F_i , $i = 1..n$ such that the union covers S . Look at the projection of these F_i .

2.2.2 use definition of a limit and the 1 norm to find δ . let $y \neq 0$, $b \neq 0$, and let $\epsilon > 0$

$$\begin{aligned} \left| \frac{x}{y} - \frac{a}{b} \right| &< \epsilon \\ \left| \frac{xb - ab + ab - ya}{yb} \right| &< \epsilon \\ |b(x - a) + a(b - y)| &< |yb|\epsilon \end{aligned}$$

Now suppose $|a| < |b|$, dividing by $|b|$ we have (a similiar statement holds for $|b| < |a|$ what is it?),

$$\left| (x - a) + \frac{a}{b}(b - y) \right| < |y|\epsilon \Rightarrow |(x - a) + (b - y)| < |y|\epsilon$$

Let $\delta = 2|y|\epsilon$, then check definition of continuity. That is

$$\forall (x, y) \in B((a, b), \delta) \Rightarrow |x/y - a/b| < \epsilon$$

2.2.4 change into polar coordinates and show that the limit is independent of θ .