Math 121B, extra problems:

Exercise 1. Suppose $T : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$, has the matrix representation:

$$[T]_\gamma = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Where γ is the standard basis. Given that the eigenvalues of T are 1 and 10.

(a) Verify that $\mathcal{E}_1$ and $\mathcal{E}_{10}$ are orthogonal.
(b) Find an ordered orthonormal basis β, such that $[T]_\beta$ is diagonal. What is $[T]_\beta$?
(c) Give an orthgonal matrix P such that P^*AP is diagonal.
(d) We know that $\mathbb{R}^3 = \mathcal{E}_1 \oplus \mathcal{E}_{10}$. So for every vector $v \in \mathbb{R}$, there is an unique $u \in \mathcal{E}_1$ and $w \in \mathcal{E}_{10}$, such that $v = u + w$, find u and w for $v = (3, 1, 1)$.

Solution: (a) Computing the eigenspace for $\lambda = 1$, we have

$$\mathcal{E}_1 = \ker \begin{vmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{vmatrix} \approx \ker \begin{vmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \text{span}((1, 0, -2), (0, 1, -2)) = \text{span}(v_1, v_2)$$

Computing the eigenspace for $\lambda = 10$, we have

$$\mathcal{E}_1 = \ker \begin{vmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{vmatrix} \approx \ker \begin{vmatrix} 4 & -5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = \text{span}((2, 2, 1)) = \text{span}(v_3)$$

Now,

$$\langle v_1, v_3 \rangle = \langle (1, 0, -2), (2, 2, 1) \rangle = 0$$
$$\langle v_2, v_3 \rangle = \langle (0, 1, -2), (2, 2, 1) \rangle = 0$$

(b) Let $w_1 = v_1$, then w_2 is given by the gram-schmit proccess,

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\|v_2\|^2} v_2 = (0, 1, -2) - \frac{\langle (0, 1, -2), (1, 0, -2) \rangle}{5} (1, 0, -2) = \frac{1}{5}(4, 5, 2)$$

all we need is something in the span of w_2, so let $w_2 = (4, 5, 2)$, then an orthonormal basis for the eigenspace $\mathcal{E}_1$ and $\mathcal{E}_{10}$ are

$$\mathcal{E}_1 = \text{span}\{\frac{1}{\sqrt{5}}(-1, 0, 2), \frac{1}{\sqrt{45}}(4, 5, 2)\} \quad \text{and} \quad \mathcal{E}_{10} = \text{span}\{\frac{1}{3}(2, 2, 1)\}$$

Now $\beta = \{hatw_1, \hat{w}_2, \hat{w}_3\}$, and so $[T]_\beta$ is given by

$$[T]_\beta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{vmatrix}$$

(c) Let $P = [\hat{w}_1, \hat{w}_2, \hat{w}_3]$

(d) want to solve the system $a_1 v_1 + a_2 v_2 + a_3 v_3 = (3, 1, 1)$

$$\left| \begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{array} \right| \approx \left| \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right|$$

and so let $u = (1, 0, -2) - (0, 1, 2) = (1, -1, 0)$, and let $w = (2, 2, 1)$, then $u + w = (3, 1, 1)$.

Exercise 2. Let V be an inner product space, and W be a finite dimensional subspace of V. Prove that x belongs to $W^\perp$ if x is orthogonal to every member of a basis β of W.

Solution: Let $\beta = \{w_i\}$ for $i = 1..n$. Let x be orthogonal to every element of β. Now we can write $w \in W$

$$w = \sum_{i=1}^{n} \alpha_i w_i$$

Now

$$\langle x, w \rangle = \langle x, \sum_{i=1}^{n} \alpha_i w_i \rangle = \sum_{i=1}^{n} \overline{\alpha}_i \langle x, w_i \rangle = 0$$

since $\langle x, w_i \rangle = 0$ for all i, therefore $x \in W^{\perp}$.

Exercise 3. Let V, be an inner product space, W be a subspace of V, and $T \in \mathcal{L}(V)$. Suppose W has a basis of eigenvectors of T^*. Prove that $W^{\perp}$ is T-invariant.

Solution: Let $v \in \beta$, and let $x \in W^{\perp}$. Now

$$\langle Tx, v \rangle = \langle x, T^* v \rangle = \langle x, \lambda v \rangle = \overline{\lambda} \langle x, v \rangle = 0$$

This is for all elements $v \in \beta$. This implies that if $x \in W^{\perp}$, then $Tx \in W^{\perp}$, i.e. $W^{\perp}$ is T-invariant.

Exercise 4. Let V, be an inner product space, W be a subspace of V and let β be an orthonomral basis for W.
(a) Prove that for any vector $v \in V$, there is a vector $u \in W$, such that $v - u \in W^{\perp}$.
(b) Prove that the u in part (a) is unique.

Solution: (a) let $\beta = \{w_i\}$ and define u as follows

$$u = \sum_{j=1}^{n} \langle (v, w_j) w_j$$

Now

$$\begin{aligned}
\langle v - u, w_i \rangle &= \langle v, w_i \rangle - \langle u, w_i \rangle \\
&= \langle v, w_i \rangle - \langle \sum_{j=1}^{n} \langle (v, w_j) w_j, w_i \rangle \\
&= \langle v, w_i \rangle - \sum_{j=1}^{n} \langle v, w_j \rangle \langle w_j, w_i \rangle \\
&= \langle v, w_i \rangle - \langle v, w_i \rangle \ = \ 0
\end{aligned}$$

Therefore $v - u \in W^{\perp}$

(b) Suppose $u, u' \in W$ are two use vectors, then $u - u' \in W$, and

$$u - u' = (v - u) - (v - u') \Rightarrow u - u' \in W^{\perp} \Rightarrow u - u' \in W \cap W^{\perp} \Leftrightarrow u = u' = 0 \ \therefore u = u'$$

hence we have