ODE/PDE qual study guide

James C. Hateley

1. Ordinary Differential Equations

Exercise 1.1. *Consider the nonlinear system:*

$$\begin{cases} \dfrac{dx_1}{dt} = x_2^3 + x_1 \\ \dfrac{dx_2}{dt} = -x_1x_2^2 + 2x_2 \end{cases}.$$

Show that the solutions of this system exist for $t \geq 0$, that is, prove that the solutions do not blow up in finite time.

Proof: Let $\langle \cdot, \cdot \rangle$ be the standard inner product, and consider $\langle \mathbf{x}, f(\mathbf{x}) \rangle$.

$$\begin{aligned} \langle \mathbf{x}, f(\mathbf{x}) \rangle &= x_1(x_2^3 + x_1) + x_2(-x_1x_2^2 + 2x_2) \\ &= x_1^2 + 2x_2^2 \\ &\leq 2(x_1^2 + x_2^2) \\ &= 2\|\mathbf{x}\|^2 \end{aligned}$$

Now consider the following $\langle x, x \rangle$:

$$\begin{aligned} \langle x, x \rangle = \langle x, x - f(x) + f(x) \rangle &= \langle x, x - f(x) \rangle + \langle x, f(x) \rangle \\ &\leq \langle x, x - f(x) \rangle + 2\langle x, x \rangle \end{aligned}$$

Which implies that $\langle x, f(x) - x \rangle \leq \langle x, x \rangle$, and also

$$\begin{aligned} \langle x, f(x) - x \rangle &= \langle x + f(x) - f(x), f(x) - x \rangle \\ &= \langle x - f(x), f(x) - x \rangle + \langle f(x), f(x) - x \rangle \\ &= -\langle f(x) - x, f(x) - x \rangle + \langle f(x), f(x) \rangle - \langle f(x), x \rangle \\ &\leq \langle x, x \rangle \end{aligned}$$

and so we have $\|f(x)\|^2 - \|f(x) - x\|^2 \leq 2\|x\|^2$, hence $\|f(x)\|^2 \leq 2\|x\|^2$. Now a system will blow up in finite time if there is a $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that as $|t - t_0| < \epsilon$ $\|\mathbf{X}(\mathbf{x}, t)\| > M$ for any $M \in \mathbb{R}$. The function $f(\mathbf{x})$ is continuous, so given an inital condition $\mathbf{X}(t_0) = \mathbf{x}_0$, where $t_0 > 0$ there exists a δ, η such that for all $B_\delta(t_0)$, $B_\eta(x_0)$ a solution exists. By the above inequality and for this solution we have $\|f(x)\| \leq 2\|x\|$ for all $x \in \mathbb{R}^2$, hence there cannot exist and M such that $\|f(x)\| > M$ for all $x \in B\eta(x_0)$, i.e., the solution does does not blow up in finite time for $t \geq 0$.

Theorem: (Peano Existence) Assume that $\mathbf{X}(\mathbf{x}, \mathrm{t})$ is continuous in the closed domain $\|\mathbf{x} - c\| \leq K$, $|t - a| \leq T$. Then an initial value problem for an ODE has at least one solution in the interval $|t - a| = \min\{T, K/M\}$, where

$$M = \sup_{\substack{\|x-c\|=K \\ |t-a|=T}} \|\mathbf{X}(\mathbf{x}, t)\|$$

Exercise 1.2. *Consider the eigenvalue problem* $\begin{cases} -u'' + \frac{1}{x+1}u' = \lambda u, & 0 \leq x \leq 1 \\ u(0) = u(1) = 0. \end{cases}$.

(a) Show that the eigenvalues are real and explain in what sense the eigenfunctions corresponding to distinct eigenvalues are orthogonal.
(b) Show that for fixed λ, the eigenvalue problem cannot have two independent eigenfunctions.

Proof: Write as a sturm liouville problem, the eigenvectors are $\perp$ w.r.t the weighted inner product $\langle \cdot, \cdot \rangle_{r(x)}$, $(L[u(x)] = r(x)\lambda u(x))$

Exercise 1.3. *Determine all critical points of the following system and find their type and stability:*

$$\begin{cases} \frac{dx}{dt} = 1 - xy \\ \frac{dy}{dt} = x - y^3 \end{cases} .$$

Proof: The critical points are determined by setting the system equal to zero. Doing this we have the following two points $(\pm 1, \pm 1)$. To determine their stability first recall that if A is a 2×2 matrix, then the characteristic equation can be computed in terms of $\text{tr}(A)$ and $\det(A)$ by solving $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$. The critical equation for determining stability is the equation $4\det(A) = \text{tr}^2(A)$, and the information can be seen in Figure 1. For the first point $(1,1)$ the Jacobian of the system is $J(1,1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

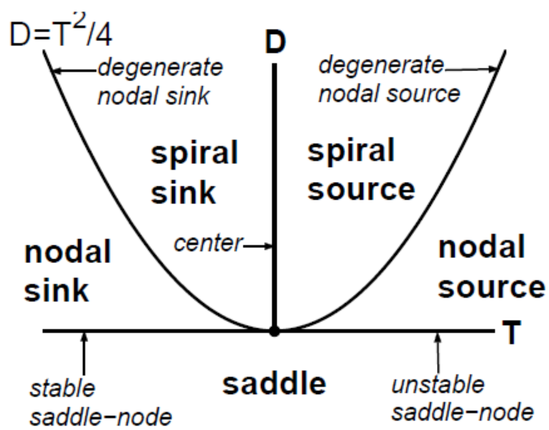

FIGURE 1.

and so $\text{tr}(J(1,1)) = -4, \det(J(1,1)) = 4$. From this we have the point $(1,1)$ as a degenerate nodal sink. For the point $(-1,-1)$ we have $J(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ and so $\text{tr}(J(1,1)) = -2, \det(J(1,1)) = -4$, which is a saddle node.

Exercise 1.4. *Using the initial value problems*

$$\frac{dx}{dt} = x^2(t),\ x(0) = 1 \qquad \textit{and} \qquad \frac{dx}{dt} = 1 + x^2(t),\ x(0) = 1$$

and a comparison principle, give a lower and upper bounds of T, where $[0,T]$ is the interval of existence of the solutions of the initial value problem

$$\frac{dx}{dt} = t^2 + x^2(t),\ x(0) = 1.$$

Proof: Solve the first and second equations to get

$$x(t) = \frac{1}{1-t}, \quad x(t) = \tan\left(t + \frac{\pi}{4}\right)$$

Now we have

$$x^2 \le t^2 + x^2 \le 1 + x^2, \text{ for } t \in [0,1]$$

if $y(t)$ is the solution we must have

$$\frac{1}{1-t} \le y(t) \le \tan\left(t + \frac{\pi}{4}\right)$$

Hence we have a lower bound at $t = 1$ and an upper bound at $t = \frac{\pi}{4}$

Exercise 1.5. *Show that the planar system*

$$\begin{cases} \dfrac{dx}{dt} = (1-x^2-y^2)x - y \\ \dfrac{dy}{dt} = x + (1-x^2-y^2)y \end{cases}$$

has a unique closed orbit γ, and show that γ is a periodic attractor.

Proof: First notice if you use linearization that you get $(0,0)$ as a source. Consider this equation in polar coordinates.

$$r^2 = x^2 + y^2 \quad \Rightarrow \quad rr' = xx' + yy'$$

Plugging the equation for x' and y' into this we have

$$rr' = x\left((1-x^2-y^2)x - y\right) + y\left(x + (1-x^2-y^2)y\right)$$

simplifying and solving we have

$$r' = (1-r^2)r$$

For θ we have the conversion factor

$$\tan(\theta) = \frac{y}{x} \quad \Rightarrow \quad \sec^2(\theta) = \frac{y'x - x'y}{y^2}$$

Again plugging x' and y' into this equation and simplifing we arrive at

$$\theta' = 1$$

Now $r = 1$ is a critical point and if $0 < r < 1$ we have $r' > 0$ and if $r > 1$ we have that $r' < 0$. Hence $r = 1$ is a stable limit cycle.

Theorem: (Poincare' Bendixson) Given a differentiable real dynamical system defined on an open subset of the plane, then every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either a fixed point, a periodic orbit, or a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.

Exercise 1.6. *Consider the Cauchy problem*

$$\begin{cases} y' = \sin(xy) \\ y(0) = y_0 > 0. \end{cases}$$

(a) Show that it has a unique solution, ϕ, defined in $\mathbb{R}$.
(b) Show that $\phi(x) > 0$, $\forall x \in \mathbb{R}$.
(c) Show that $\phi(x) = \phi(-x)$, $\forall x \in \mathbb{R}$.

Proof: First notice that $|\sin(x_1y_1) - \sin(x_2y_2)| < 2$ forall ordered pairs in $\mathbb{R}^2$. Now consider the interval $x \in [-a,a]$, we have

$$\left|\frac{\sin(xy_1) - \sin(xy)}{y_1 - y}\right|$$

passing to the limit we have as $y_1 \to y$

$$\frac{d}{dy}\sin(xy) = x\cos(xy) \le x \le a \text{ for } x \in [-a,a]$$

Hence we have

$$|\sin(xy_1) - \sin(xy_2)| < 2a|y_1 - y_2|$$

Hence the function is Lipschitz continuous on $[-a,a]$ in y. The function is clearly continous in x, hence by Picard-Lindelof theorem there exists a unique solution $\phi(x)$ defined on $[-a,a]$. Since a was arbitrary we can make it as large as we need, hence the solution is on all of $\mathbb{R}$.

Exercise 1.7. *Consider the family of Cauchy problems:*

$$CP(n) = \begin{cases} y' = f(y) \\ y(x_0) = y_{0,n} \end{cases}$$

in $\mathbb{R}^m$, where f is Lipschitz continuous in $\mathbb{R}^m$ and $x_0 \in [a,b]$. Assume that each problem has a solution $\phi_n : [a,b] \to \mathbb{R}^m$. Note that we assume that all solutions are defined in the same interval $[a,b]$. Assume that $y_{0,n} \to y_0$ as $n \to \infty$.

(a) *Prove that $\{\phi_n\}$ forms a Cauchy sequence in the space of continuous functions $C([a,b];\mathbb{R}^m)$, with the sup norm.*

(b) *Conclude that $\{\phi_n\}$ converges uniformly to a continuous funtion $\phi \in C([a,b];\mathbb{R}^m)$, and that ϕ solves the Cauchy problem*

$$CP(\infty) = \begin{cases} y' = f(y) \\ y(x_0) = y_0 \end{cases}$$

Proof: Let $M = \max\{f(x,y) : (x,y) \in [a,b] \times D\}$ where D is compact inside $\mathbb{R}^n$. Using the Picard iterations we have

$$\phi_{n,k}(x) = y_{0,n} + \int_{x_0}^{x} f(y_k(t))\ dt$$

where $\phi_{n,k}(x) \rightrightarrows \phi_n(x)$ Now since each solution converges we have that $|\phi_{n,k}(x) - \phi_{n,l}(x)| < \epsilon$ if $l,k > N$ for some $N \in \mathbb{N}$

$$\begin{aligned} |\phi_{m,k} - \phi_{n,l}| &= |\phi_{m,k} - \phi_{n,l} + \phi_{m,l} - \phi_{m,l}| \\ &\leq |\phi_{m,k} - \phi_{m,l}| + |\phi_{n,l} - \phi_{m,l}| \\ &\leq \epsilon + |y_{0,n} - y_{0,m}| + \int_{x_0}^{x} |f(y_k(t)) - f(y_l(t))|\ dx \\ &\leq 3\epsilon \end{aligned}$$

The above can be choosen small enought if k,l,m,n are large enough by assumptions of the problem. This implies the sequence of solutions converge uniformly, since the values does not depend on x, since the integral over a large set of something small can be made small. All of the hypothesis satisfy the the Picard-Lindelof theorem, hence the solution is unique.

Exercise 1.8. *In this problem you are asked to find, amoung all function $y(x)$, for which $y''(x)$ exists and is continuous at each point in the interval $0 \leq x \leq 1$, the one which minimizes the integral.*

$$I[y] = \int_0^1 [(y'(x))^2 + y^2(x) - xy(x)]\ dx$$

Note that no restrictions are imposed on the boundary conditions of the function $y(x)$.

(a) *Write the corresponding Euler-Lagrange equation including boundary conditions.*

(b) *Prove that if $y \in C^2([0,1])$ is a solution of the Euler-Lagrange equation, then*

$$I[y+w] \geq I[y], \quad \forall w \in C^1([0,1])$$

and conclude that y is a minimizer of I

(c) *Solve the Euler-Lagrange equation to find the minimizer.*

Proof: Recall that the Euler-Lagrange equation is looking for the minimizer of the following

$$S(y) = \int_a^b L(x,y,y')\ dx$$

with the boundary conditions that $y(a) = c, y(b) = d$. The equation is given by

$$L_y - \frac{d}{dx} L_{y'} = 0$$

Using the problem given we have

$$2y + x - \frac{d}{dx}(2y') = 0 \quad \Rightarrow \quad y'' - y' + \frac{x}{2} = 0$$

The characteristic equation is $r^2 - r = 0$ so the homogenious solution is $y_H(x) = c_1 e^x + c_2$. Using the method of undetermed coefficents we have

$$y_P = a + bx + cx^2 \quad \Rightarrow \quad y_P = -\frac{x}{4} - \frac{x^2}{2}$$

Hence $y(x) = y_H(x) + y_P(x)$ Let $f_\epsilon(x) = y(x) = \epsilon w(x)$. Show that

$$S(f_\epsilon) = \int_0^1 F(x, f_\epsilon, f_\epsilon)'$$

minimizes when $\epsilon = 0$ by computing $\frac{d}{d\epsilon}$ and showing that y is the minimizer.

Exercise 1.9. *Consider the boundary value problem*

$$-\frac{\partial^2 x(t)}{\partial t^2} + \lambda x(t) = g(t), \quad x'(0) = x'(\pi) = 0.$$

(a) *if $g = 0$, for which values of the parameter $\lambda \in \mathbb{R}$ does the problem have a nontrivial solution?*
(b) *Given $g \in C([0, \pi])$, under what conditions on g and λ does the problem have a solution? Under what conditions on g and λ is the solution unique? Justify your answer.*

Proof: Let $g(t) = 0$. Look at the 3 cases $\lambda = 0$, $\lambda > 0$, $\lambda < 0$. The only case that is not trivial is if $\lambda = \mu^2 > 0$ in which the eigenvalues and eigenvectors are

$$\lambda_n = \frac{2n-1}{2}, \quad \phi_n(x) = b_n \sin(\lambda_n x), \quad n \in \mathbb{Z}$$

Now if $g(t) \neq 0$, use variation of parameters with fundamental solutions $y_1 = e^{\imath \mu t}, y_2 = e^{-\imath \mu t}$. Compute the Wronskian, given by $W = y_1 y_2' - y_2 y_1' = -2\imath u$, then need the general solution as $y = a(x)y_1 + b(x)y_2$. Where

$$a(x) = -\int \frac{1}{w} u_2 g(x)\ dx \quad b(x) = -\int \frac{1}{w} u_1 g(x)\ dx$$

Exercise 1.10. *Consider the equation of the undamped pendulum*

$$\frac{\partial^2 x(t)}{\partial t^2} + \sin(x(t)) = 0.$$

(a) *Write the equation as a first order system, and find the fixed points.*
(b) *show that the system obtained in part (a) is Hamiltonian, and find the Hamiltonian $H(x, y)$.*
(c) *Describe the stability of the stationary solutions in (a).*
(d) *Draw a phase portrait of the solutions near the stationary points.*

Proof: Written as a system we have

$$y' = -\sin(x), x' = y$$

Now the critical points are $y = 0$ and $x = k\pi, k \in mZ$. Integrating each equation we have the Hamiltonian,

$$H(x, y) = \frac{y}{2} - \cos(x)$$

This comes from

$$H(x, y) = \int x'\ dy - \int y'\ dx$$

Exercise 1.11. *Let $f_1(x) = e^x$ and $f_2(x) = \sin(x)$*

(a) *Find $a_1(x)$ and $a_2(x)$ such that for $|x| < \delta$ with $\delta \in (0, \pi/4)$ the funcitons f_1, f_2 form a fundamental set of solutions of the second order linear equation*

$$\frac{\partial^2 f(x)}{\partial x^2} + a_1(x)\frac{df(x)}{dx} + a_2(x)f(x) = 0, \quad |x| < \delta$$

(b) *Can one extend a_1, a_2 to the closed interval $[-\pi/4, \pi/4]$ such that f_1, f_2 still staify the property in part (a) in this interval? Justify your answer.*

Proof: text

Exercise 1.12. *Define*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Compute the fundamental matrix of A.
(b) Explain how to use the fundamental matrix of A to solve the initial problem

$$x'' + x = 0, \qquad x(0) = x_0, x'(0) = x_1$$

Proof: text

2. Partial Differential Equations

Exercise 2.1. *Consider the second order PDE*

$$\partial_x^2 u(x,t) - 3\partial_{xt} u(x,t) - 4\partial_t^2 u(x,t) = 0, \qquad x \in \mathbb{R}, t \geq 0$$

(a) Factor the differential operator as the product of two first order differential operators.
(b) Using part (a) find the general solution of the equation.
(c) Find the solution of the equation satisfying the initial data

$$u(x,0) = x, \quad \partial_t u(x,0) = 1.$$

Proof: text

Exercise 2.2. *Find the explicit solution of the initial value problem*

$$\begin{cases} u_t + uu_x = 0, & t \geq 0, x \in \mathbb{R} \\ u(0,x) = x, & x \in \mathbb{R} \end{cases}$$

Proof: text

Exercise 2.3. *Find the solution of the boundary value problem for the Laplace equation*

$$\begin{cases} \Delta u(x,y) = 0, & 1 < x^2 + y^2 < 4 \\ u(x,y) = -5, & x^2 + y^2 = 1 \\ u(x,y) = 0, & x^2 + y^2 = 4 \end{cases}$$

Proof: text

Exercise 2.4. *Find the solution of the initial boundary value problem*

$$\begin{cases} u_t(x,y) = u_{xx}(t,x), & (t,x) \in \mathbb{R}^+ \times (0,1) \\ u(t,0) = 0, u(t,1) = 1 & t \in \mathbb{R}^+ \\ u(0,x) = \sin\left(\frac{\pi x}{2}\right), & x \in [0,1] \end{cases}$$

and find $\lim_{t\to\infty}$.

Proof: text

Exercise 2.5. *Consider the Dirichlet problem for the Laplace equation in the unit Ball $B_1(0)$ of $\mathbb{R}^2$ written in polar coordinates*

$$\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}u = 0, \qquad u(1,\theta) = f(\theta), \qquad r \in (0,1), \theta \in [0, 2\pi)$$

whose solution is given by the formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{)}^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t} f(t)\ dt.$$

If $f(\theta) = (\theta - \pi)^2$, answer the following questions:

(a) What is the value of u at the origin?

(b) Show that for any $(r,\theta) \in (0,1) \times [0, 2\pi)$ it follows that

$$u(0)\frac{1-r}{1+4} < u(r,\theta) < u(0)\frac{1+r}{1-r}$$

(c) Can one find $(r_0, \theta_0) \in (0, 1/2] \times [0, 2\pi)$ such that $u(r_0, \theta_0) < 1$. Justify your answer.

Proof: text

Exercise 2.6. *Let $u(x,t)$ be a solution of the boundary value problem*

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t > 0, x \in (-1,1) \\ u(x,0) = 1 - x^2, u(t,1) = 1 & x \in [-1,1] \\ u(0,t) = u(1,t) = 0, & t > 0 \end{cases}$$

Prove that $u(x,t) = u(-x,t)$ for all $(x,t) \in [-1,1] \times [0,\infty)$ **Proof:** text

Exercise 2.7. *Find the solution of the initial value problem*

$$\begin{cases} u_t - c^2 u_{xx} = 2t, & t > 0, x \in \mathbb{R}, \\ u(x,0) = x^2, u_t(x,0) = 1 & x \in \mathbb{R}. \end{cases}$$

Assume that $c \neq 0$, justify your answer.

Proof: text

Exercise 2.8. *Solve the initial value problem*

$$\begin{cases} x\partial_x u + \partial_y u = y, & y > 0, x \in \mathbb{R}, \\ u(x,0) = x^2, & x \in \mathbb{R}. \end{cases}$$

Justify your answer.

Proof: text

Exercise 2.9. *(a) Suppose $u \in C^2(\mathbb{R}^2)$ is a solution of the wave equation*

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \qquad (t,x) \in \mathbb{R}^2$$

such that $u(0,x) = u_t(0,x) = 0$ for all $|x| \leq 1$. Prove that $u(t,x) = 0$ in the region

$$\mathcal{R} = \{(t,x) \in \mathbb{R}^2 : |x+t| \leq 1, |x-t| \leq 1\}.$$

(b) *Suppose that* $u_j \in C(\mathbb{R}^2), j = 1, 2,$ *are solutions of the wave equation for all* $(t, x) \in \mathbb{R}^2$ *such that*

$$u_1(0, x) = u_2(0, x) \quad \text{and} \quad (u_1)_t(0, x) = (u_2)_t(0, x),$$

for all $x \leq 1$*. Prove that* $u_1(t, x) = u_2(t, x)$ *in* $\mathcal{R}$*.*

Proof: *text*

Exercise 2.10. *Let* $f \in C(\mathbb{R})$ *and* $a \in C^1(\mathbb{R})$*. Solve the initial value problem*

$$\begin{cases} u_t(t, x) + a'(t)u_x(t, x) = 0, & (t, x) \in \mathbb{R}^2 \\ u(0, x) = f(x), fx \in \mathbb{R} \end{cases}$$

Proof: text

Exercise 2.11. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary.*

(a) *Prove that if the eigenvalue prolem*

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a nonzero solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *then the eigenvalue* λ *must be strictly postive.*

(b) *Suppose that* $\overline{\lambda} > 0$ *is a value for which the eigenvalue problem in part (a) has a nonzero solution* $\overline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$*. Given* $f \in C(\overline{\Omega})$*, prove that if the problem*

$$-\Delta u = \overline{\lambda} u + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$*, then*

$$\int_\Omega \overline{u}(x) f(x)\, dx = 0.$$

Proof: text

Exercise 2.12. *Suppose that* $\phi \in C([0, 1])$ *with* $\phi(0) = \phi(1) = 0$*. Solve the initial boundary value problem.*

$$\begin{cases} u_t(t, x) = u_{xx}(t, x) + \sin(t)\sin(\pi x), & (t, x) \in \mathbb{R}^+ \times (0, 1) \\ u(t, 0) = u(t, 1) = 0, & t \in \mathbb{R}^+ \\ u(0, x) = \phi(x), & x \in [0, 1]. \end{cases}$$

Your solution will involve the Fourier sine coefficients $\{c_k\}_{k=1}^\infty$ *of* ϕ

Proof: text