Complex qual study guide

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General Complex Analysis

Problem: Let $p(z)$ be a polynomial. Suppose that $p(z) \neq 0$ for $\Re(z) > 0$. Prove that $p'(z) \neq 0$ for $\Re(z) > 0$.

Solution: Let $p(z)$ be such a polynomial. Suppose that $a_i \in \mathbb{C}$ are the zeros of p(z). Then we have

$$p(z) = c\prod_{i=1}^{n}(z - a_i), \quad p'(z) = c\sum_{i=1}^{n}\prod_{i\neq j}(z - a_j)$$

Now the quotient is given by

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n}\frac{1}{z - a_i}$$

If $p'(z) = 0$, then the sum above must be equal to zero. Now if $\Re(a_i) \leq 0$ this implies that $\Re(z_0 - a_i) > 0$, by the assumption of the hypothesis. So there is a w such that $w = z_0 - a_i = x + \imath y$, where $x > 0$. Now we have

$$\frac{1}{z_0 - a_i} = \frac{1}{w} = \frac{1}{x + \imath y} = \frac{x - \imath y}{x^2 + y^2} \quad \Rightarrow \quad \Re\left(\frac{1}{z_0 - a_i}\right) > 0$$

Since each term in the sum $\dfrac{1}{z - a_i}$ has a positive real part we have $p'(z) \neq 0$ for $\Re(z) > 0$ $\square$.

Problem: Describe those polynomials $a + bx + cy + dx^2 + exy + fy^2$ with real coefficients that are the real parts of analytic functions on $\mathbb{C}$.

Solution: let $u(x, y) = a + bx + cy + dx^2 + exy + fy^2$. $u(x, y)$ being the real part of a holomorphic function implies that $u(x, y)$ is harmonic. So $\Delta u(x, y) = 0$ and

$$\partial_{xx}u = 2d, \partial_{yy}u = 2f \quad \Rightarrow \quad d + f = 0 \ \square$$

Problem: Prove or disprove that there exists an analytic function $f(z)$ in the unit disc $D(0, 1)$ such that

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}, \quad \forall n \in \mathbb{N}$$

Solution: Suppose there exists such a function with the above property. Since $f(z)$ is analytic, it is uniquely determined by a cauchy sequence. Now $\{1/n\}$ is clearly cauchy, hence

$$f\left(\frac{1}{n}\right) = \frac{1}{n^3} \quad \Rightarrow \quad f(z) = z^3$$

but clearly $f(z)$ is an odd function and hence $f\left(-\dfrac{1}{n}\right) \neq \dfrac{1}{n^3}$. Therefore there is no such function that satisfies the hypothesis $\square$

Problem: Let $p(z)$ be a polynomial such that all the roots of $p(z)$ lie in $D(0, 1)$. Prove that the roots of $p'(z)$ lie in $D(0, 1)$.

Proof: We will prove something stronger, that all the roots of $p'(z)$ lie inside the convex hull of the roots of $p(z)$. Suppose all the roots of $p(z)$ lie in $D(0, 1)$, Let $\{a_i\}$ be the set of roots of $p(z)$ including multiplicities, then we have

$$p(z) = c\prod_{i=1}^{n}(z - a_i), \quad |a_i| \in D(0, 1), c \in \mathbb{C}$$

Taking the logarithmic derivative we have

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - a_i}$$

In particular, if z is a zero of $p'(z)$ and still $p(z) \neq 0$, then

$$\sum_{i=1}^{n} \frac{1}{z - a_i} = 0$$

which implies

$$\sum_{i=1}^{n} \frac{\overline{z} - \overline{a_i}}{|z - a_i|^2} = 0$$

This may also be written as

$$\left(\sum_{i=1}^{n} \frac{1}{|z - a_i|^2} \right) \overline{z} = \sum_{i=1}^{n} \frac{1}{|z - a_i|^2} \overline{a_i}.$$

Taking their conjugates, we see that z is a weighted sum with positive coefficients that sum to one. If $p(z) = p'(z) = 0$, then $z = 1z + 0a_i$, and is still a convex combination of the roots of $p(z)$. Since the unit disk is a convex hull of the roots of p(z), then all roots of p'(z) are inside the unit disk. □

Problem: Prove or disprove that there is a sequence of analytic polynomials $\{p_n(z)\}, n \in \mathbb{N}$, so that $p_n(z) \to \bar{z}^4$ as $n \to \infty$ uniformly for $z \in \partial D(0, 1)$.

Solution: The statement is not true. Suppose that there exists such a sequence of analytic polynomials such that $p_n(z) \to \bar{z}^4$. Then for all n we have $\frac{d}{d\bar{z}} p_n(z) = 0$ since $p_n(z)$ is analytic. However $\frac{d}{d\bar{z}} \bar{z}^4 = 4\bar{z}^3 \neq 0$ for all $z \in \mathbb{C}$. Clearly $0 \not\to 4\bar{z}^3$ for all $z \in \mathbb{C}$ □

Problem: The Bernoulli polynomials $\phi_n(z)$ are defined by the expansion

$$\frac{e^{tz} - 1}{e^t - 1} = \sum_{n=1}^{\infty} \frac{\phi_n(z)}{n!} t^{n-1}$$

Prove the following two statements:
a) $\phi_n(z + 1) - \phi_n(z) = nz^{n-1}$

Proof: Let $B(z) = \frac{e^{tz} - 1}{e^t - 1}$, then $B(z + 1) - B(z) = e^{tz}$. Now by definition of $B(z)$ we have the following expansion

$$\sum_{n=1}^{\infty} \frac{\phi_n(z + 1)}{n!} t^{n-1} - \sum_{n=1}^{\infty} \frac{\phi_n(z)}{n!} t^{n-1} = \sum_{n=0}^{\infty} \frac{(tz)^n}{n!}$$

reindexing we have

$$\sum_{n=0}^{\infty} \frac{\phi_n(z + 1)}{n!} t^{n-1} - \sum_{n=0}^{\infty} \frac{\phi_n(z)}{n!} t^{n-1} = \sum_{n=0}^{\infty} \frac{(tz)^{n-1}}{(n - 1)!}$$

Hence we have $\phi_n(z + 1) - \phi_n(z) = (n + 1)z^{n-1}$ □

b) $\frac{\phi_{n+1}(n + 1)}{n + 1} = \sum_{k=1}^{n} k^k$

Proof: By the previous part we have

$$\begin{aligned} \phi_{n+1}(n + 1) - \phi_{n+1}(n) &= (n + 1)n^n \\ \phi_{n+1}(n) - \phi_{n+1}(n - 1) &= (n + 1)(n - 1)^n \\ &\vdots \\ \phi_{n+1}(2) - \phi_{n+1}(1) &= (n + 1)1^n \end{aligned}$$

Rearraging and using the recursive relation above we have

$$\begin{aligned}
\phi_{n+1}(n+1) &= (n+1)n^n + \phi_{n+1}(n) \\
&= (n+1)n^n + (n+1)(n-1)^n + \phi_{n+1}(n-1) \\
&\vdots \\
&= (n+1)\sum_{k=0}\sum_{k=1}^{n} k^k \\
\Rightarrow \quad \frac{\phi_{n+1}(n+1)}{n+1} &= \sum_{k=1}^{n} k^k \ \square
\end{aligned}$$

Complex Integration

Computer the area of the image of the unit disk $D = \{z : |z| < 1\}$ under the map $f(z) = z + \frac{z}{2}$.

Solution: Denote $\Omega = f(D)$, and let $d\sigma$ denote the surface measure, then an integral for the surface area is given by ;

$$\int_{\Omega} d\sigma = \int\int_{D} J(u,v)\ dA$$

Now let $f(z) = f(x,y) = u(x,y) + \imath v(x,y)$, then we have

$$f(x,y) = x + \imath y + \frac{(x+\imath y)^2}{2} = x + \frac{x^2 - y^2}{2} + \imath\,(y + xy)$$

Computing $J(u,v)$, we have

$$J(u,v) = \begin{vmatrix} 1+x & -y \\ y & 1+x \end{vmatrix} = (1+x)^2 + y^2$$

Converting to polar coordinates we come to the integral

$$\int_{\Omega} d\sigma = \int_0^{2\pi}\int_0^1 1 + 2r\cos(\theta) + r^2)r\ dr d\theta = \frac{3\pi}{2}\ \square$$

Problem: Evaluate the integral:

$$\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx$$

Solution: Consider the following contour Γ:

$$\Gamma = \begin{cases} \gamma_1 := \quad t & t \in [-R, -1/R] \\ \gamma_2 := e^{\imath t}/R & t \in [\pi, 2\pi] \\ \gamma_3 := \quad t & t \in [1/R, R] \\ \gamma_4 := Re^{\imath t} & t \in [0, \pi] \end{cases}$$

Now our function has a removable singularity at $x = 0$, so consider the following

$$f(x) = \frac{1 - e^{2\imath x}}{2x^2} \quad \Rightarrow \quad \Re(f(x)) = \frac{1 - \cos(2x)}{2x^2} = \frac{\sin^(x)}{x^2}$$

Now for the integral around Γ we have

$$\int_{\Gamma} f(z)\ dz = 2\pi\imath\,\mathrm{Res}(f(z)) = 2\pi\imath \lim_{z\to 0} \frac{d}{dz} z^2 f(z) = 2\pi\imath \lim_{z\to 0} -\imath e^{2\imath z} = 2\pi\imath(-\imath) = 2\pi$$

Now for the integral on γ_1 we have

$$\int_{\gamma_1} f(z)\ dz = \frac{1}{2}\int_{-R}^{-1/R} \frac{1 - e^{2\imath t}}{t^2}\ dt \quad \Rightarrow \quad \frac{1}{2}\int_{-\infty}^{0} \frac{1 - e^{2\imath t}}{t^2}\ dt \text{ as } R \to \infty$$

For the integral on γ_2 we have

$$\begin{aligned}\int_{\gamma_2} f(z)\ dz &= \frac{1}{2}\int_{\pi}^{2\pi} \frac{\left(1-e^{2\imath e^{\imath t}/R}\right)\imath e^{\imath t}/R}{e^{2\imath t}/R^2} \\ &= \frac{\imath}{2}\int_{\pi}^{2\pi} \frac{1-e^{2\imath e^{\imath t}/R}}{e^{2\imath t}/R}\end{aligned}$$

Now letting $R\to\infty$ and using L'Hospitals rule we have

$$\frac{\imath}{2}\int_{\pi}^{2\pi} \frac{-e^{2\imath e^{\imath t}/R}\,(2\imath e^{\imath t}/R)\,(\imath)}{\imath e^{\imath t}/R} = \int_{\pi}^{2\pi}\ dt = \pi$$

For the integral on γ_3 we have

$$\int_{\gamma_3} f(z)\ dz = \frac{1}{2}\int_{1/R}^{R} \frac{1-e^{2\imath t}}{t^2}\ dt \quad \Rightarrow \quad \frac{1}{2}\int_0^{\infty} \frac{1-e^{2\imath t}}{t^2}\ dt \text{ as } R\to\infty$$

Now for γ_4 we have

$$\begin{aligned}\int_{\gamma_4} f(z)\ dz &= \frac{1}{2}\int_0^{\pi} \frac{1-Re^{2\imath t}}{R^2e^{2\imath t}} R\imath e^{\imath t}\ dt \\ &= \frac{\imath}{2}\int_0^{\pi} \frac{1-Re^{2\imath t}}{Re^{\imath t}}\end{aligned}$$

Putting this all together we have

$$2\pi = \pi + \frac{1}{2}\int_{-\infty}^{0} \frac{1-e^{2\imath t}}{t^2}\ dt + \frac{1}{2}\int_0^{\infty} \frac{1-e^{2\imath t}}{t^2}\ dt$$

Taking real parts we have

$$\pi = \int_{-\infty}^{0} \frac{\sin^2(t)}{t^2}\ dt + \int_0^{\infty} \frac{\sin^2(t)}{t^2}\ dt = 2\int_0^{\infty} \frac{\sin^2(t)}{t^2}\ dt$$

Hence we have $\displaystyle\int_0^{\infty} \frac{\sin^2(t)}{t^2}\ dt = \frac{\pi}{2}$ □

Taylor and Laurent series

Problem: Find the largest disc centered at 1 in which the Taylor series for

$$\frac{1}{1+z^2} = \sum_{k=1}^{\infty} a_k(z-1)^k$$

will converge.

Solution: The singularities of $\frac{1}{1+z^2}$ occur at $\pm\imath$. First consider the taylor series centered at 0;

$$\frac{1}{1+z^2} = \sum_{k=1}^{\infty}(-1)^k z^{2k}$$

instead of recomputing the coeffients a_k and taking the limsup, notice the raduis of converge for the series at 0 is 1. Since the series must avoid the sigularities the radius will be the distance from the center to the closest singularity, i.e. $r = \inf\{|1-\imath|, |1+\imath|\} = \sqrt{2}$ □

Problem: Find the raduis of convergence for the series:

$$\sum_{n=1}^{\infty} \frac{z^{2n}}{n!} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{z^{n!}}{2n}$$

Solution: For the first one we have

$$\limsup_{n\to} \left|\frac{z^{2n}}{n!}\right|^{1/n} < 1 \quad \Rightarrow \quad |z|^2 < \lim_{n\to\infty} \sup n!^{1/n}$$

Now $\lim_{n\to\infty} n!^{1/n} = \infty$, hence $|z| < \infty$.

For the second series we have

$$\limsup_{n\to} \left|\frac{z^{n!}}{2n}\right|^{1/n} < 1 \quad \Rightarrow \quad |z| < \left(\lim_{n\to\infty} \sup (2n)^{1/n}\right)^{1/(n-1)!} = 1$$

Hence the series converges for $|z| < 1$ □

Problem: Let f be a non-constant entire function. Prove that if $\lim_{|z|\to\infty} |f(z)| = \infty$, then $|f|$ must be a polynomial.

Solution: Consider $g(z) = f\left(\frac{1}{z}\right)$, then $\lim_{z\to 0} g(z) = \infty$. Now suppose that $g(z)$ has a pole of order k and consider the Laurent expansion:

$$g(z) = \frac{a_{-k}}{z^k} + \frac{a_{-k+1}}{z^{k-1}} + \cdots + a_0 + a_1 z + \cdots \quad \Rightarrow \quad z^k g(z) = z^k \sum_{n=-k}^{\infty} a_n z^n$$

Now $|z^k g(z)| \to c = f(0)$ as $|z| \to \infty$. This implies by continuity that $|z^k g(z)| \leq (|c|+1)z^k$ for large z. Hence $z^k g(z)$ is a polynomial of at most degree k. Now we have:

$$g(z)z^k = \sum_{n=0}^{k} a_n z^n \quad \Rightarrow \quad f\left(\frac{1}{z}\right) = g(z) = \sum_{n=0}^{k} \frac{a_n}{z^{k-n}}$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{k} a_n z^{k-n}$$

Problem: Show that for $R > 0$, there is N_R such that when $n > N_R$, the function

$$P_n(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} \neq 0, \quad \forall\ |z| \leq R.$$

Solution: First notice that $P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ and that $P_n(z) \to e^z$ uniformly as $n \to \infty$ on compact sets of $\mathbb{C}$. Fix $R > 0$

$$\forall \epsilon > 0\ \exists N_R\ s.t. \left|\sum_{k=0}^{n} \frac{z^k}{k!} - \sum_{k=0}^{m} \frac{z^k}{k!}\right| = \left|\sum_{k=m}^{n} \frac{z^k}{k!}\right| \leq \epsilon, \quad \forall n > m > N_R.$$

This implies that

$$\left|e^z - \sum_{k=0}^{n} \frac{z^k}{k!}\right| < \epsilon, \quad \forall n > N_R$$

which implies that

$$1 \leq |e^z| < \epsilon + \left|\sum_{k=0}^{n} \frac{z^k}{k!}\right|, \quad \forall n > N_R, \forall z \in \overline{D(0,R)}$$

$$\therefore \forall R > 0\ \exists N_R\ s.t. \sum_{k=0}^{n} \frac{z^k}{k!} \neq 0, \quad \forall n > N_R\ \square$$

Problem: Let $f(z)$ be analytic on $\mathbb{C} - \{1\}$ and have a simple pole at $z = 1$ with residue λ. Prove that for every $R > 0$,

$$\lim_{n\to\infty} R^n \left|(-1)^n \frac{f^{(n)}(2)}{n!} - \lambda\right| = 0$$

Proof: Since $f(z)$ has simple pole at one we have the Laurent expansion.

$$f(z) = \frac{\lambda}{z-1} + \sum_{n=0}^{\infty} a_n (z-1)^n$$

Define $g(z) = f(z) - \dfrac{\lambda}{z-1}$, then $g(z)$ is an entire function. Now for $|z-2| < 1$ f(z) has the taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(z-2)^2$$

Also we have the geometric series for $\frac{\lambda}{z-1}$

$$\frac{\lambda}{z-1} = \frac{\lambda}{1+(z-2))} = \lambda \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$

This implies that the series for $g(z)$ about 2 is

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(z-2)^n - \lambda \sum_{n=0}^{\infty} (-1)^n (z-2)^n = \sum_{n=0}^{\infty} (z-2)^n \left(\frac{f^{(n)}(2)}{n!} - \lambda(-1)^n \right)$$

Now for $|z-2| < 1$ we have that $(z-2)^n \left| \dfrac{f^{(n)}(2)}{n!} - \lambda(-1)^n \right| \to 0$. But since $g(z)$ is entire we this holds for any $z \in \mathbb{C}$ Hence for any $R > 0$ we have

$$R^n \left| \frac{f^{(n)}(2)}{n!} - \lambda(-1)^n \right| \to 0 \text{ as } n \to \infty$$

So the result is shown. □

Problem: Find the radius of convergence R_1 of the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

and show the series converges uniformly on $\overline{D}(0, R_1)$. What is the radius of convergence R_2 of the derivative of this series? Does it converge uniformly on $\overline{D}(0, R_2)$?

Solution: Denote the above series by $f(z)$. By taking limsup we find that the raduis of convergence is 1. Let $z \in \overline{D}(0,1)$, then we have

$$\left| \sum_{n=1}^{\infty} \frac{z^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

hence the series converges uniformly on $\overline{D}(0,1)$. Now the derivative $f'(z)$ of the series will converge on the unit disc by Abel's theorem. But will not converge uniformly on $\overline{D}(0,1)$, since if $z = 1$, the series diverges □.

Problem: Let $f(z)$ be analytic in the punctured unit disk $U_0 = \{z : 0 < |z| < 1\}$ such that thre is a positive interger n with $|f^n(z)| \leq |z|^{-n}$ for all $z \in U_0$. Show that $z = 0$ is a removable singularity for $f(z)$

Solution: Let $g(z) = z^n f^{(n)}(z)$, then $g(z) \leq 1$ for all $z \in D(0,1)$. This implies that $z = 0$ is a removable singularity of $g(z)$. Now consider the Laurent series expansion for $g(z)$ inside $D(0,1)$.

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad \Rightarrow \quad f^{(n)}(z) = \sum_{n=0}^{\infty} a_n z^{k-n}$$

Now $f(z)$ has the laurent expansion

$$f(z) = \sum_{n=-k}^{\infty} b_n z^k \quad \Rightarrow \quad f^{(n)}(z) = \sum_{n=0}^{\infty} b_n \frac{(-1)^n (n+k-1)!}{(k-1!)} z^{n-k}$$

but since the Laurent expansion is unique, we must have $b_k = 0$ for all $k < 0$. Which implies that $f(z)$ has a Taylor expansion about $z = 0$. Therefore $f(z)$ has a removeable singularity at $z = 0$. $\square$

Problem: Let $f(z)$ be analytic in the disk $U = \{|z| < 1\}$, with $f(0) = f'(0) = 0$. Show that $g(z) = \sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ defines an analytic function on U. Moreover, show that the above function $g(z)$ satisfies

$$g(z) = f(z) \sum_{n=1}^{\infty} \frac{1}{n^2}$$

if and only if $f(z) = cz^2$.

Solution: Consider the Taylor expansion for $f(z)$ with the conditions $f(0) = f'(0) = 0$, this implies that

$$f(z) = z^2 \sum_{k=0}^{\infty} a_k z^k = z^2 h(z)$$

for some holomorphic function $h(z)$. Plugging in z/n we have

$$f\left(\frac{z}{n}\right) = \frac{z^2}{n^2} \sum_{k=0}^{\infty} a_k \left(\frac{z^k}{n^k}\right) = \frac{z^2}{n^2} h\left(\frac{z}{n}\right)$$

Since $f(z/n)$ is analytic for all n it suffices to show that the series $\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ converges normally on U. Let K be a compact set of U Since $h(z)$ is analytic on K it is continuous. Hence $h(z)$ attains it's maximum on K, denote this value as M. Now we have

$$\left| f\left(\frac{z}{n}\right) \right| = \left| \frac{z^2}{n^2} h\left(\frac{z}{n}\right) \right| \leq \frac{|z|^2}{n^2} M$$

Hence if $z \in K$ then the seris $\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ convergies absolutely on K. Hences by the Weierstrass M-test $\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ converges uniformly on K, which implies the series converges normally since K was an arbitrary closed set in U. Therefore $g(z)$ is analytic in U, since it's the normal limit of analytic function on U.

For the second part, if $f(z) = cz^2$, then we have

$$\sum_{n=0}^{\infty} f\left(\frac{z}{n}\right) = \sum_{n=0}^{\infty} c\frac{z^2}{n^2} = cz^2 \sum_{n=0}^{\infty} \frac{1}{n^2} = f(z) \sum_{n=0}^{\infty} \frac{1}{n^2}$$

On the other hand, suppose

$$g(z) = f(z) \sum_{n=0}^{\infty} \frac{1}{n^2}.$$

consider the talyor expansion for $f(z)$, plugging this in we have

$$g(z) = \left(\sum_{k=0}^{\infty} a_k z^k\right) \left(\sum_{n=0}^{\infty} \frac{1}{n^2}\right) = \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{\infty} \frac{1}{n^k}\right) z^k$$

but

$$g(z) = \sum_{n=0}^{\infty} f\left(\frac{z}{n}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \frac{z^k}{n^k} = \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{\infty} \frac{1}{n^2}\right) z^k$$

Since the power series for an analytic function is unique, this implies that $a_k = 0$ for all $k \neq 2$. Therefore $f(z) = a_2 z^2$ $\square$

Applications of Cauchy's Interal formula

Problem: Let $U \subset \mathbb{C}$ be a connected open set, and γ be a closed curve in U. Suppose that for any function $f(z)$ holomorphic on U we have

$$\oint f(z)dz = 0.$$

Does it imply that γ is homotopic to a constant curve?

Solution: No γ does not have to a constant curve, consider the function $f(z) = z^{-1}$, on the punctured disk $U = D(2,1) - \{2\}$. Then $f(z)$ is holomorphic on U, now fix $r \in (0,1)$ and let $\gamma = re^{\imath t} + 2$ for $t \in [0, 2\pi]$, then by Cauchy's theorem we have

$$\int_\gamma f(z)dz = 0.$$

but $re^{\imath t} + 2$ is clearly not a constant curve. $\square$

Problem: Let $f(z)$ be entire holomorphic function on $\mathbb{C}$ such that $|f(z)| \leq |\cos(z)|$. Prove $f(z) = c\cos(z)$ for some constant c.

Solution: Consider $g(z) = \dfrac{f(z)}{\cos(z)}$, then $|g(z)| \leq 1$, hence $g(z)$ is a bounded function. Define $\widehat{g}(z)$ as follows:

$$\widehat{g}(z) = \begin{cases} g(z) & \text{if } \cos(z) \neq 0 \\ \lim_{z \to w} g(z) & \text{if } \cos(w) = 0 \end{cases}$$

Then $\widehat{g}(z)$ is a bounded entire function. Hence by Lioville's theorem it must, i.e. $\widehat{g}(z) = c$ for some $c \in \mathbb{C}$. It follows from the definition of $g(z)$ that $f(z) = c\cos(z)$ $\square$

Problem: Prove that there is no entire analytic function such that

$$\bigcup_{n=0}^{\infty} \{z \in \mathbb{C} : f^{(n)}(z) = 0\} = \mathbb{R}$$

Solution: First there exists an N such that $S = \{z \in \mathbb{C} : f^{(N)}(z) = 0\}$ is dense in $\mathbb{R}$, if not then $\mathbb{R}$ is a countable union of nowhere dense sets, which is a contradiction to the Baire Cateogry theorem. Now let $z_0 \in S$, then for every $\epsilon > 0$, the disc $D(z_0, \epsilon)$ contains infintely many points in S. Now let $\zeta \in D(z_0, \epsilon)$ such that $\zeta \notin S$, now by Cauchy estimates we have

$$|f^{(N)}(\zeta)| \leq \frac{N!}{2\pi} \int_{|z+z_0|=\epsilon} \frac{|f(z)|}{|z-\zeta|}^{N+1} dz$$

Now consider the change of variables $w = z - \epsilon$, then

$$\frac{N!}{2\pi} \int_{|z+z_0|=\epsilon} \frac{|f(z)|}{|z-\zeta|}^{N+1} dz = \frac{N!}{2\pi} \int_{D(w,\epsilon)} \frac{|f(w+\epsilon|)}{|w|^{N+1}} \to 0 \text{ as } \epsilon \to 0.$$

This implies that $f^{(N)}(\zeta) = 0$, hence $\zeta \in S$ which is a contradiction to $\zeta \notin S$. Therefore there cannot exist such a function $\square$

Problem: Find all entire functions $f(z)$ on $\mathbb{C}$ satisfying

$$|f(z)| \leq |z|e^x, \quad z = x + \imath y \in \mathbb{C}$$

Solution: First notice the following:

$$|z|e^x = |ze^x| = |ze^x e^{\imath y}| = |ze^z|$$

Let $g(z) = \dfrac{f(z)}{ze^z}$, then $g(z)$ is a bounded function since $|g(z) = \dfrac{|f(z)|}{|ze^z|} < 1$. Hence the discontinuity at $z = 0$ is removable. Define

$$\widehat{g}(z) = \begin{cases} \frac{f(z)}{ze^z} & z \neq 0 \\ \lim_{z\to 0} \dfrac{f(z)}{ze^z} & z = 0 \end{cases}$$

Now since $f(z)$ and ze^z are entire we have $\widehat{g}(z)$ as a bounded entire function. Hence by louvilles theorem $\widehat{g}(z) = k \in \mathbb{C}$. So we have $f(z) \leq kze^z$, where $|k| \leq 1$ □.

Problem: Complete the following problems:
a) State the Lioville's theorem

Lioville's theorem states that a bounded entire function is constant.

b) Prove the Lioville's theorem by calculating the following integral

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz$$

and taking the limit $R \to \infty$.

Solution: Suppose that $f(z)$ is a bounded entire function, that is, there exists $M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$. Fix $R > 0$, now for $a, b \in D(0, R)$ the integral is bounded by;

$$\left| \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz \right| \leq \frac{2\pi RM}{(R-|a|)(R-|b|)} \to 0 \text{ as } R \to \infty$$

Now by direct computation we have

$$\begin{aligned} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz &= 2\pi\imath\left(\operatorname{Res} f(a) + \operatorname{Res} f(b)\right) \\ &= 2\pi\imath \frac{f(b)-f(a)}{b-a} = 0 \end{aligned}$$

This implies that $f(b) = f(a)$ for all $a, b \in \mathbb{C}$, hence $f(z)$ is constant.

Problem: Find the number of zeros of the function $f(z) = 2z^5 + 8z - 1$ in the annulus $1 < |z| < 2$.

Solution: Let $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$, then $\partial D = \{z \in \mathbb{C} : |z| = 1 \text{ or } |z| = 2\}$. Now consider the function $g(z) = 2z^5 + 8z$. Then $|f(z) - g(z)| = 1$ on ∂D. Now

$$|g(z)| = |2z^5 + 8z| = |z||2z^4 + 8|, \text{ on } \partial D \text{ and } 1 = \min_{z\partial D} |z| \leq |g(z)|$$

So we have $|f(z) - g(z)| = 1 < 1 + |f(z)|$ on ∂D. Also both $f(z)$ and $g(z)$ are holomorphic on $\overline{D}$. Hence By Rouche's theorem $f(z)$ and $g(z)$ have the same number of zeros in D. Now $g(z) = z(2z^4 + 8)$, which implies $z = 0$ and $z^4 = -4$. So the set of zeros that lie in D are

$$z = 4^{1/4}e^{2\imath\pi k/4}, \quad k = 0, 1, 2, 3$$

So $g(z)$ has 4 roots in $D \therefore f(z)$ has 4 roots in D □

Problem: Find all roots of the equation $2z + \sin(z) = 0$ in the unit disc.

Solution: Clearly $z = 0$ is a root, to show that this is the only root consider $f(z) = 2z$, $g(z) = \sin(z)$. Let $z \in \partial D(0, 1)$, now by convexity of e^z we have

$$|g(z)| = \frac{|e^{\imath z} - e^{\imath z}|}{2} \leq \frac{e^{|z|} + e^{-|z|}}{2} = \frac{e}{2} + \frac{1}{2e} < 2$$

This implies that

$$|g(z)| < 2 = 2|z| = |f(z)| \quad \forall z \in \partial D(0,1)$$

Hence by Rouche's theorem $f(z)$ and $f(g) + g(z)$ have the same number of roots in $D(0,1)$. $2z$ has 1 root in $D(0,1)$, therefore $2z + \sin(z)$ has 1 root in $D(0,1)$, which is $z = 0$ $\square$

Problem: If $f(z)$ is an entire function satisfying the estimate

$$|f(z)| \leq 1 + |z|^{\sqrt{2010}} \quad \forall z \in \mathbb{C}$$

Show that $f(z)$ is a polynomial and determine the best upperbound for the degree of $f(z)$.

Solution: First observe that $44 < \sqrt{2010} < 45$. Let $R > 0$ and consider the Cauchy estimate for $f^{(n)}(0)$ on $D(0,R)$.

$$\left| f^{(n)}(0) \right| \leq \frac{n!(1 + R^{\sqrt{2010}})}{R^n}$$

Now if we consider the Taylor expansion for $f(z)$ about $z = 0$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C} \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

If $n > \sqrt{2010}$, let $\alpha = n - \sqrt{2010} > 0$ then we have the estimates let

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{n!} \cdot \frac{n!(1 + R^{\sqrt{2010}})}{R^n} = \frac{1}{R^n} + \frac{1}{R^\alpha} \to 0 \text{ as } R \to \infty$$

Hence we have $a_n = 0$ for all $n > \sqrt{2010}$, which implies that $a_n = 0$ for all $n \geq 45$. Therefore $f(z)$ is a polynomial of degree at most 44. $\square$

Problem: Show that $f(z) = \alpha e^z - z$ has only one zero in $U = \{|z| < 1\}$ if $|\alpha| < 1/3$ and no zeros if $|\alpha| > 3$.

Solution: For $|\alpha| < 1/3$, let $g(z) = -z$ then we have

$$|f(z) - g(z)| = |\alpha e^z| \leq |\alpha| e^x < \frac{1}{3} e < 1 = |g(z)| \text{ on } \partial U$$

Thus by Rouche's thoerem $f(z)$ and $g(z)$ have the same number of zeros in U. Therefore $f(z)$ has exactly one zero in U since $g(z) = -z$ has one zero.

For $|\alpha| > 3$, let $h(z) = \alpha e^z$, then we have

$$|f(z) - g(z)| = |z| = 1 \leq \frac{3}{e} < \frac{|\alpha|}{e} \leq |\alpha e^z| = |h(z)| \text{ on } \partial U$$

Thus by Rouche's thoerem $f(z)$ and $h(z)$ have the same number of zeros in U. Therefore $f(z)$ has no zeros in U, since $h(z)$ no roots. $\square$

Problem: Show that there is a holomorphic function defined in the set

$$\Omega = \{z \in \mathbb{C} : |z| > 4\}$$

Whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a holomophic function on Ω whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)}?$$

Solution: Let γ be a closed curve lying outside of Ω. Now if there exists such a function $F(z)$, such that

$$F'(z) = f(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

then the following condition should be satisfied

$$\int_\gamma F'(z) = 0$$

It suffices to show this is true for $\gamma = re^{\imath t}$, where $r > 0$. Now $F'(z)$ has 3 simple poles $\{1, 2, 3\}$ lying inside of γ. Hence we have

$$\begin{aligned}\int_\gamma f(z) &= 2\pi\imath\,(\text{Res}\,f(1)\,\text{Res}\,f(2) + \text{Res}\,f(3)) \\ &= 2\pi\imath\left(\frac{1}{2} - 2 + \frac{3}{2}\right) = 0\end{aligned}$$

Hence by Morera's there does exists such a function $F(z)$ such that $F'(z) = f(z)$ □

Now consider the function

$$g(z) = \frac{z^2}{(z-1)(z-2)(z-3)}.$$

Using the same ideas above, we have

$$\begin{aligned}\int_\gamma g(z) &= 2\pi\imath\,(\text{Res}\,f(1)\,\text{Res}\,f(2) + \text{Res}\,f(3)) \\ &= 2\pi\imath\left(\frac{1}{2} - 4 + \frac{9}{2}\right) \neq 0\end{aligned}$$

Hence there cannot exist a homomorphic function $G(z)$ such that $G'(z) = g(z)$ □

Find the integral

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1}\, dz$$

Solution: Let $f(z) = z^8 - 2z + 1$ and $g(z) = z^8$, then we have

$$|f(z) - g(z)| = |-2z + 1| \leq 2|z| + 1 = 5 \quad \text{on} \quad \partial D(0, 2)$$

Also we have

$$5 < 2^8 = |z|^8 = |g(z)| \quad \text{on} \quad \partial D(0, 2)$$

Hence by Rouche's theorem $f(z)$ and $g(z)$ have the same number of zeros including multiplicties in $D(0, 2)$. Since $g(z)$ has 8 zeros, $f(z)$ has 8 zeros in $D(0, 2)$. Now observe

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1}\, dz = \frac{1}{2}\int_{|z|=2} \frac{f'(z)}{f(z)}\, dz$$

and $f(z)$ is entire and non-vanishing on $\partial D(0, 2)$, since all of the roots of $f(z)$ lie inside $D(0, 2)$. Hence by the argument principle we have

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1}\, dz = \frac{1}{2}(8) = 4 \;\square$$

Normal Families

Problem: Let $\mathcal{F}$ be a family of holomorphic functions on the unit disk D for which there exists $M > 0$ such that

$$\int_D |f(z)| dxdy \leq M, \quad \forall f \in \mathcal{F}.$$

Show that $\mathcal{F}$ is a normal family.

Solution: We want to show $f \in \mathcal{F}$ is bounded. Consider the following construction, fix $r \in (0, 1)$

$$\overline{D}(0, 1) \subset \bigcup_{z \in D(0,1)} D(z, r)$$

Since $\overline{D}(0,1)$ is compact there exists a finite number of $\{z_k\}$ such that

$$D(0,1) \subset \overline{D}(0,1) \subset \bigcup_{k=1}^{n} D(z_k, r)$$

Let $\epsilon = \inf\{r, |z_k - \partial D(0,1)\}$, now for any $f \in \mathcal{F}$ we have

$$\begin{aligned} f(z_k) &= \frac{1}{\pi\epsilon^2} \int_{D(z_k,\epsilon)} f(z)\ dA \\ \Rightarrow \quad |f(z_k)| &\leq \frac{1}{\pi\epsilon^2} \int_{D(z_k,\epsilon)} |f(z)|\ dA \\ &\leq \frac{1}{\pi\epsilon^2} \int_{D(0,1)} |f(z)|\ dA \\ &\leq \frac{M}{\pi\epsilon^2} = M_\epsilon \end{aligned}$$

Since $D(z_k, \epsilon) \subset D(0,1)$. Now this is for all z_k and for all $f \in \mathcal{F}$. Let $M_0 = \max M_\epsilon$. where the sup is taken over all possible finite covers of $\bigcup_{z \in D(0,1)} D(z,r)$. Then we have $|f(z)| \leq M_0$. Hence $\mathcal{F}$ is a bounded family. Therefore by Montel's theorem

$$\forall \{f_k\} \subset \mathcal{F}\ \exists f_{k_j}\ s.t.\ f_{k_j} \xrightarrow{u} f_0$$

Where f_0 is holomorphic in $D(0,1)$, i.e. $\mathcal{F}$ is a normal family. □

Problem: Consider the family of functions $\{f_\alpha\}_{\alpha \in A}$ that is holomorphic on a domain U. Suppose that for all $z \in U$, and for all $\alpha \in A$ we have $\Re(f(z) \neq (\Im(z)(f(z))^2$. Prove that $\{f_\alpha\}$ is a normal family.

Solution: Consider the following two domains $U_1 := \{z : x < y^2\}$ and $U_2 := \{z : x > y^2\}$. By the Riemann open mapping theorem, there exists maps ϕ_1 such that $\phi_1 : U_1 \to D(0,1)$ and ϕ_2 such that $\phi_2 : U_2 \to D(2,1)$. Now consider the following function:

$$h(z)_\alpha = \begin{cases} \phi_1 \circ f_\alpha, & f_\alpha \in U_1 \\ \phi_2 \circ f_\alpha, & f_\alpha \in U_2 \end{cases}$$

then h_α is holomorphic in U and bounded hence by Montels theorem h_α is a normal family □. Therefore $\{f_\alpha\}$ is a normal family.

Problem: Let $\mathcal{F} = \{f_\alpha\}$ be a family of holomorhiphc functions on $D(0,1)$ and for all $z \in D(0,1)$

$$|f'(z)|(1 - |z|^2) + |f(0)| \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

Proof: Let $\epsilon > 0$ and consider $D(0,r)$ where $r = 1 - \epsilon$. Now for $z \in \overline{D}(0,r)$ we have

$$|f'(z)| \leq \frac{1 - |f(0)|}{1 - |z|^2}$$

Using the triangle inequality and integrating we have

$$f(z) = \left| \int f'(z)\ dz \right| \leq \int |f'(z)| dz \leq \int \frac{1 - |f(0)|}{1 - |z|^2} dz < \infty$$

this is valid for all $z \in \overline{D}(0,r)$ and for any $f(z) \in \mathcal{F}$. Hence by Montel's theorem $\mathcal{F}$ is a normal family. □

Problem: Let Ω be a bounded domain in $\mathbb{C}$, and let $\{f_j\}, j \in \mathbb{N}$ be a sequence of analytic functions on Ω such that

$$\int_\Omega |f_j(z)|^2 dA(z) \leq 1$$

Prove that $\{f_j\}$ is a normal family in Ω

Proof: By definition a normal family implies that for every compact subset K of Ω, there exists a subsequence $\{f_{j_k}\}$ that converges uniformly some f_0 in K. Fix a compact set $K \subset \Omega$. Let $r > 0$, now we have

$$K \subset \bigcup_{x \in K} D(x, r)$$

as an open cover for K. Since K is compact there exists finite set $\{x_i\}$, such that

$$K \subset \bigcup_{i=1}^{n} D(x_i, r)$$

Now for each x_j, we have the by the mean value theorem for holomorphic functions

$$\begin{aligned} |f_j(x_i)| &\leq \frac{1}{\pi r^2} \int_{D(x_i,r)} |f_j(x,y)| dA \\ \text{by Hölders inequality} &\leq \frac{1}{\pi r^2} \mu(D(x_i, r)) \|f_j(x,y)\|_2 \leq 1 \end{aligned}$$

Since $\mu(D(x_i, r)) = \pi r^2$ and $\|f_j(x,y)\|_2^2 \leq 1$ by the hypothesis. Now this implies that $f_j(x_i)$ is bounded for all j and x_i, hence it is uniformly bounded. Therefore by Montel's theorem there exists a subsequence $\{f_{j_k}\}$ such that f_{j_k} converges uniformly on K. Thus $\{f_j\}$ is a normal family in Ω □

Problem: (a) State the Montel Theorem for normal family.

Montel's Theorem: Let $\mathcal{F}$ be a family of holomorphic functions on an open set $U \subset \mathbb{C}$. Suppose that for each compact set $K \subset U$, there is $M = M(K)$ such that $|f(z)| \leq M$ for all $z \in K$ and all $f \in \mathcal{F}$. Then for every $\{f_\alpha\} \subset \mathcal{F}$, there is a subsequence $\{f_{\alpha_k}\}$ that converges uniformly on compact subsets of U to a holomorphic limit, in otherwords, $\mathcal{F}$ is a normal family.

(b) Let $\mathcal{F}$ be a set of holomorphic functions on the unit disk $D(0,1)$ so that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta < 1.$$

Show that $\mathcal{F}$ is a normal family.

Solution: Let $K \subset U$ be compact. Since K is compact and contained in $D(0,1)$, there is $0 < R < 1$ such that $K \subset D(0,R) \subset D(0,1)$. Define $\epsilon > 0$ as follows:

$$\epsilon = \frac{1}{2} \text{dist} \left(\partial K, \partial U \right).$$

If $z \in K$ and $f \in \mathcal{F}$, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|w-z|=\epsilon} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \oint_{|w|=R+\epsilon} \frac{f(w)}{w-z} dw, \end{aligned}$$

where the second line is by Cauchy's integral theorem. Hence we have

$$|f(z)| \leq \frac{1}{2\pi} \oint_{|w|=R+\epsilon} \frac{|f(w)|}{|w-z|} dw.$$

Now if $|w| = R + \epsilon$ and $z \in K$, we have $|w - z| \geq \epsilon$ since $|z| \leq R$. Hence $\frac{1}{|w-z|} \leq \frac{1}{\epsilon}$. Thus,

$$\begin{aligned} |f(z)| &\leq \frac{1}{2\pi\epsilon} \oint_{|w|=R+\epsilon} |f(w) dw| \\ &= \frac{1}{2\pi\epsilon} \int_0^{2\pi} |f((R+\epsilon)e^{i\theta})|(R+\epsilon) d\theta \\ &= \frac{(R+\epsilon)}{2\pi\epsilon} \int_0^{2\pi} |f((R+\epsilon)e^{i\theta})| d\theta \leq \frac{(R+\epsilon)}{2\pi\epsilon}. \end{aligned}$$

Therefore $|f(z)|$ is uniformly bounded on K. The same bound holds for all $f \in \mathcal{F}$. Since this is for any $K \subset\subset U$, by Montel's theorem $\mathcal{F}$ is a normal family. $\square$

Harmonic Functions

Problem: Let U be a bounded, connected, open subset of $\mathbb{C}$, and let f be a nonconstant continuous function on $\overline{U}$ which is holomorphic on U. Assume that $|f(z)| = 1$ for z on the boundary of U.

(a) Show that 0 is in the range of f.

Solution: By the max mod principle, the maximum of the function takes its max on the boundary, hence we know that $f(U) \subset D$. If $0 \notin f(U)$ then let $g(z) = \frac{1}{f(z)}$, then we have $|g(z)| = \frac{1}{|f(z)|} = 1$ which implies that $f(z) = 1$ for all $z \in \overline{U}$, which is a contradiction to the open mapping theorem. Hence 0 is in the range of f.

(b) Show that f maps U onto the unit disk.

Solution: Let $\alpha \in D$, set $B_\alpha = \dfrac{z-\alpha}{1-\overline{a}z}$. Consider $h(z) = B_\alpha \circ f(z)$, then $|h(z)| = 1$ on ∂U which implies that $0 \in h(U)$, by part (a), which implies that $a \in f(U)$. Hence the image of f is the unit disk.

Problem: Let $U : \mathbb{C} \to \mathbb{R}$ be harmonic. Prove or disprove each of the following.

(a) If $u \leq 0$ for all $z \in \mathbb{C}$, then u is constant on $\mathbb{C}$.

Proof: Suppose $u \geq 0$ in $\mathbb{C}$, then $u \geq 0$ on $D(0, R)$ for any $R > 0$, so by Harnacks inequality we have

$$\frac{R-|z|}{R+|z|}u(0) \leq u(z) \leq \frac{R+|z|}{R-|z|}u(0), \qquad \forall z \in D(0,R)$$

taking $R \to \infty$, we have $u(0) \leq u(z) \leq u(0)$. Therefore $u(z) = u(0)$ hence u is constant.

(b) If $u = 0$ for all $|z| = 1$, then $u(z) = 0$ for all $z \in \mathbb{C}$.

Proof: Suppose $u = 0$ for all $z \in \partial D(0,1)$, now consider $D(0,1)$, then by the maximum/minimum modulus principle we have

$$\max_{z \in \overline{D}(0,1)} u(z) \quad = \max_{z \in D(0,1)} u(z) \quad = \min_{z \in \overline{D}(0,1)} u(z) = 0 \quad \Rightarrow \quad u \equiv 0 \quad \forall z \in \overline{D}(0,1)$$

Now we have

$$0 = u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(p + re^{\imath\theta})\, d\theta$$

This implies that $u = 0$ on $D(z, r)$, for all $z \in \overline{D}(0,1)$ and for all $r > 0$. $\therefore u(z) = 0$ on $\mathbb{C}$ $\square$

(c) If $u = 0$ for all $z \in \mathbb{R}$, then $u(z) = 0$ for all $z \in \mathbb{C}$.

Solution: The statement is not true, consider $u(z) = u(x + \imath y) = y$, then $\Delta u \equiv 0$ and $u(z) = 0$ for all $z \in \mathbb{R}$ but $u \neq 0$ $\square$

Problem: Let u be a harmonic function on $\mathbb{R}^2$ that does not take zero value (i.e. $u(x) \neq 0, \forall x \in \mathbb{R}^2$). Show that u is constant.

Proof: $u(x,y) \neq 0$ implies that $u(x,y)$ is either strictly positive or strictly negative. It suffices to consider $u(x,y)$ as strictly positive, (otherwise consider $-u(x,y)$). Then there exist $f(z)$ holomorphic on $\mathbb{C}$, such that $f(z) = f(x,y) = u(x,y) + \imath v(x,y)$, where $u(x,y)$ is the given harmonic function and

$v(x,y)$ is the harmonic conjugate of $u(x,y)$. Now consider the following: $e^{-f(z)}$

$$|e^{-f(z)}| \leq |e^{-u(x,y)}| < 1$$

So $e^{-f(z)}$ is a bounded entire function. Hence by Liouville's theorem $e^{-f(z)}$ must be constant, which implies that $f(z)$ is constant and thus that $\Re(f(z)) = u(x,y)$ is constant $\square$

Problem: Let u be a positive harmonic function on the right half plane $\{\Re(z) > 0\}$, and $\lim_{r\to 0^+} u(r) = 0$. Prove that then $\lim_{r\to 0^+} u(re^{\imath\theta}) = 0$ for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution: Since u is harmonic on $R = \{z : \Re(z) > 0\}$, there exists a function v which is conjugate to u, i.e, the function $f(z) = u + \imath v$ is holomophic on R. Now $f(z)$ is continuous on this set R and $\lim_{r\to^+} f(r) = 0$. Now let $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $\rho(z) = e^{r\imath\phi}$. Now consider $g(z) = f \circ \rho \circ \rho^{-1}$, then we have

$$\lim_{r\to 0^+} g(z) = 0 \quad \to \quad \lim_{r\to 0^+} (f\circ\rho)z = \lim_{r\to 0^+} \rho(z) = 0$$

Hence we have $\lim_{r\to 0^+} u \circ \rho(z) = \lim_{r\to 0^+} u(re^{\imath\theta}) = 0$ $\square$

Problem: (a) Suppose a continuous function $u : \mathbb{C} \to \mathbb{R}$ has the following property:

$$u(x+\imath y) = \frac{1}{4}(u(x+a+\imath y) + u(x-a+\imath y) + u(x+\imath(y+a)) + u(x+\imath(y-a)))$$

for all $a \in \mathbb{R}$. Does it imply that u is harmonic?

Solution: Yes, $u(z)$ is harmonic and $u(z)$ is in the following the set $A = \{u : \mathbb{C} \to \mathbb{R} : u \text{ is continuous}\}$, check conditions. A is the space of the following polynomials $a(xy^3 - yx^3) + b(x^3 - 3xy^2) + c(y^3 - 3x^2y) + d(x^2 - y^2) + ex + fy + g$, where the letters are complex numbers.

(b) Suppose a continuous function $u : \mathbb{C} \to \mathbb{R}$ has the following property:

$$u(x+\imath y) = \frac{1}{4}\left(u(x+a+\imath y) + u(x-a+\imath y) + u(x+\imath(y+a)) + u(x+\imath(y-a))\right)$$

for all $a \in \mathbb{C}$. Does it imply that u is harmonic?

Solution: write $z = x + \imath y$ and write a in it's polar form, then for any $a \in \mathbb{C}$ we have the following:

$$u(z) = \frac{1}{4}(u(z+ae^{i\theta}) + u(z+ae^{i(\theta+\pi)} + u(z+re^{i(\theta+\pi/2)}) + u(z+re^{i(\theta+3\pi/2)})$$

by integrating both sides with respect to θ from $0 to 2\pi$ we have

$$2\pi u(z) = \int_0^{2\pi} u\left(z + ae^{\imath\theta}\right)\, d\theta$$

so $u(z)$ has the mean value property, hence $u(z)$ is harmonic $\square$

Problem: Let u and v be real-valued harmonic functions on the whole complex plane such that

$$u(z) \leq v(z), \quad z \in \mathbb{C}$$

Find the relation between u and v.

Solution: Let $h(z) = v(z) - u(z)$, then $\Delta h(z) = \Delta v(z) - \Delta u(z) \equiv 0$ on $\mathbb{C}$. Let $R > 0$. Then by the Harnack inequality, if $|z| < R$, as $h(z)$ is real-valued harmonic on $\overline{D}(0,R) \subset \mathbb{C}$, $0 \leq h(z)$ on $\mathbb{C}$,

$$h(0) \cdot \frac{R-|z|}{R+|z|} \leq h(z) \leq h(0) \cdot \frac{R+|z|}{R-|z|}.$$

Now fix $z \in \mathbb{C}$. Then for all $R > |z|$, the above holds, and so

$$\lim_{R\to\infty} h(0) \cdot \frac{R-|z|}{R+|z|} \leq h(z) \leq \lim_{R\to\infty} h(0) \cdot \frac{R+|z|}{R-|z|}.$$

Hence we have $h(0) \le h(z) \le h(0)$, which implies that $h(z) = h(0)$, $\forall z \in \mathbb{C}$. Thus

$$v(z) - u(z) = v(0) - u(0) \quad \Rightarrow \quad v(z) = u(z) + v(0) - u(0)$$
$$\therefore \quad v(z) = u(z) + \alpha \text{ for some } \alpha \in \mathbb{C} \ \square$$

Remark: Harnack's inequality: Let u be a nonnegative, harmonic function on a neighborhood of $\overline{D}(0,R)$. Then, for any $z \in D(0,R)$

$$u(0) \cdot \frac{R - |z|}{R + |z|} \le u(z) \le u(0) \cdot \frac{R + |z|}{R - |z|}.$$

Problem: Prove or disprove each of the statements:

(a) If f is a function on the unit disk D such that $f^2(z)$ is analytic on D, then f itself is analytic.

Solution: This statement is false. Let $f(z) = \sqrt{z}$, then $f^2(z) = z$ which is holomorphic on $D(0,1)$, but $f(z)$ is not holomorphic at $z = 0$.

(b) If $f(z)$ is a continuously differentiable function on D, and if $f^2(z)$ is analytic on D, then $f(z)$ itself is analytic.

Proof: Let $f(z) = f(x,y) = u(x,y) + \imath v(x,y)$ where u, v are harmonic. Then $f^2(z) = u^2(z) - v^2(z) + 2\imath u(z)v(z)$. Define g(z) and h(z) as follows:

$$g(z) := \Re(f^2(z)) = u^2(z) - v^2(z)$$
$$h(z) := \Im(f^2(z)) = 2u(z)v(z)$$

Now since $f^2(z)$ is holomorphic we know that $f^2(z)$ satisfies the Cauchy-Riemann equations, i.e.,

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

Computing the above we have

$$\frac{\partial g}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = \frac{\partial h}{\partial y} = 2u\frac{\partial v}{\partial y} + 2v\frac{\partial u}{\partial y}$$

this implies that

$$u\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = v\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{1}$$

Computing the other equality we have

$$\frac{\partial g}{\partial y} = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} = -\frac{\partial h}{\partial x} = -2u\frac{\partial v}{\partial x} - 2v\frac{\partial u}{\partial x}$$

which implies that

$$u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = v\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) \tag{2}$$

Solving the above system of equations in (1) and (2) implies that for all $z \in D(0,1)$ either,

$$u^2 + v^2 = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The former case implies that $u = v = 0$, which implies $f(z) = 0$ and thus $f(z)$ is analytic in $D(0,1)$. The latter case implies $f(z)$ satisfies the Cauchy-Riemann equations, and thus $f(z)$ is analytic in $D(0,1)$. Either way $f(z)$ is analytic in $D(0,1)$. $\square$

Suppose a function $f : \overline{D}(0,1) \to \mathbb{C}$ is continuous and holomorphic in D. Suppose also that for any $z \in \partial D$ we have $\Re f(z) = (\Im f(z))^2$. Prove that $f(z)$ is constant.

Proof: Let $f(z) = f(x,y) = u(x,y) + \imath v(x,y)$, where u and v are harmonic functions. From the hypothesis we know that $u = v^2$. Computing the partials we have

$$u_{xx} = 2vv_{xx} + 2v_x^2 \quad u_{yy} = 2vv_{yy} + 2v_y^2 =$$

adding and factoring we have

$$0 = u_{xx} + u_{yy} = 2v(v_{xx} + v_{yy}) + 2(v_x^2 v_y^2) = 2(v_x^2 v_y^2)$$

this implies that $v_x = v_y = 0$ for all $z \in \partial D$, hence v and u are constant on ∂D. Then by uniquness of the taylor expansion for $f(z) = u(x,y) + \imath v(x,y)$, $f(z)$ must be constant as well. □

Conformal Mappings

Important Conformal maps

· $\text{Aut}(\mathbb{C}) = \{f(z) : f(z) = az + b, a \neq 0\}$
· $\text{Aut}(D(0,1) = \{f(z) : f(z) = e^{\imath\theta}\frac{z-a}{1-\bar{a}z}, a \in D(0,1), \theta \in [0,2\pi]\}$
· $\text{Aut}(D(0,1) - \{0\}) = \{f(z) : f(z) = ze^{\imath\theta}, \theta \in [0,2\pi]\}$
· $\text{Aut}(\mathbb{C} \cup \{\infty\}) = \{f(z) : f(z) = \frac{az+b}{cz+d}, ad - bc \neq 0\}$
· $\text{Biholo}(\{z : \Im(z) > 0\}, D(0,1)) = \frac{z-\imath}{z+\imath}$ (Cayley transform)
· $\text{Biholo}(D(0,1), \{z : \Im(z) > 0\}) = \imath\frac{1+z}{1-z}$ (inverse Cayley transform)
· $\text{Biholo}(\{z : \Im(z) > 0\}, \{z : \Im(z) > 0, \Re(z) > 0\}) = \sqrt{z}$ (applies for the half disk as well)
· Two annli $\{z : r_1 < |z| < r_2\}, \{z : s_1 < |z| < s_2\}$ are conformally equivalent iff $r_2/r_1 = s_2/s_1$
· $\text{Biholo}(\{z : 0 < \Im(z) < \imath, D(0,1)) = e^z$
· $\text{Biholo}(\{z : -\pi/4 < \Re(z) < \pi/4, D(0,1)) = \tan(z)$
· $\text{Biholo}(\{z : 1/2 < \Re(z), D(1,1)) = \frac{1}{z}$
· $\text{Biholo}(\{z : 0 < \Im(z), \{0 < \Im(z), 0 < \Re(z) < \infty\}) = \sin^{-1}(z)$

Problem: Find explicitly a conformal mapping ϕ which maps the strip

$$\left\{z \in \mathbb{C} : \frac{1}{3} < \Re(z) < 1\right\}$$

to the unit disk.

Solution: Let $\Omega = \left\{z \in \mathbb{C} : \frac{1}{3} < \Re(z) < 1\right\}$, define ϕ_1 on Ω by

$$\phi_1 : z \to \frac{3}{2}\left(z - \frac{1}{3}\right)$$

Then $\phi_1(\Omega) = \Omega_1 := \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$. Now define ϕ_2 on Ω_1 by

$$\phi_2 : z \to \imath\pi z$$

Then $\phi_1(\Omega_1) = \Omega_2 := \{z \in \mathbb{C} : 0 < \Im(z) < \pi\}$. Now define ϕ_3 on Ω_2 by

$$\phi_3 : z \to e^z$$

Then $\phi_3(\Omega_2) = \Omega_3 := \{z \in \mathbb{C} : 0 < \Im(z)\}$. Finally define ϕ_4 on Ω_3 by

$$\phi_4 : z \to \frac{z - \imath}{z + \imath}$$

Then $\phi_4(\Omega_3) = D(0,1)$. Hence the composition

$$\phi := \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1 = \frac{e^{\imath\pi\left(\frac{3}{2}z - \frac{1}{2}\right)} - \imath}{e^{\imath\pi\left(\frac{3}{2}z - \frac{1}{2}\right)} + \imath}$$

maps Ω conformally onto $D(0,1)$ □

Problem: Find explicitly a conformal mapping of the domain

$$U = \{z \in \mathbb{C} : |z| < 1, \Re(z) > 0, \Im(z) > 0\}$$

to the unit disk.

Solution: First consider the map
$$\phi_1(z) = z^2$$
this takes the quarter disk into the upper half disk. Then the map
$$\phi_2(z) = \imath\frac{1+z}{1-z}$$
takes the half disk into the upper half plane and finally the cayley transform
$$\phi_3(z) = \frac{z-\imath}{z+\imath}$$
which takes the upper half plane into the unit disk. Therefore the map
$$(\phi_3 \circ \phi_2 \circ \phi_1)z = \frac{(1+z)-\imath(1-z)^2}{(1+z)+\imath(1-z)^2}$$
maps the quarter disk U conformally to the unit disk $\square$

Problem: Find explicitly a conformal mapping of G onto the unit disk, where
$$G = \left\{z = x + \imath y; |z| < 1 \text{ and } y > -1/\sqrt{2}\right\}$$

Solution: Consider the map $\phi_1(z)$ defined as
$$\phi_1(z) = \frac{\sqrt{2}z + i + 1}{\sqrt{2}z + i - 1}.$$
This sends $\frac{1}{\sqrt{2}}(-1-i)$ to 0, $-\frac{1}{\sqrt{2}}i$ to -1, and $\frac{1}{\sqrt{2}}(1-i)$ to ∞. Then let $G_1 := \phi_1(G)$, so we have
$$G_1 = \{z = re^{\imath\theta} : 0 < r < \infty,\ \pi/4 < \theta < \pi\}$$
Now ϕ_1 maps G into G_1 conformally. Next consider the map $\phi_2(z)$ defined by
$$\phi_2(z) = e^{-\frac{\pi}{4}i}z$$
Then let $G_2 := \phi_2(G_1)$, so we have
$$G_2 = \{z = re^{\imath\theta} : 0 < r < \infty,\ 0\theta 3\pi/4\}$$
Then ϕ_2 maps G_1 into G_2 conformally. Now define $\phi_3(z)$ by
$$\phi_3(z) = z^{\frac{4}{3}}$$
Then let $G_3 := \phi_3(G_2)$, so we have
$$G_3 = \{z = x + \imath y : y > 0\}$$
Then ϕ_3 maps G_2 into G_3 conformally. Finally define ϕ_4 as the Cayley transform , which maps the upper half plane into the unit disk. So define $\phi(z)$ as
$$\phi(z) = (\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1)(z) \Rightarrow \phi(z) = \frac{\left(e^{-\frac{\pi}{4}i}\left(\frac{\sqrt{2}z+i+1}{\sqrt{2}z+i-1}\right)\right)^{\frac{4}{3}} - i}{\left(e^{-\frac{\pi}{4}i}\left(\frac{\sqrt{2}z+i+1}{\sqrt{2}z+i-1}\right)\right)^{\frac{4}{3}} + i}.$$
Then $\phi(z)$ maps the set G into the unit disk conformally. $\square$

Schwarz Reflection Priciple

Problem: Let $L \subset \mathbb{C}$ be the line $L = \{x + \imath y : x = y\}$. Assume that f is an entire function, such that for any $z \in L$, $f(z) \in L$. Assume that $f(1) = 0$. Prove that $f(\imath) = 0$.

Proof: Because 1 and $\imath$ have symmetry about L, we want to consider the Schwarz reflection principle. First consider the following change of coordinates:
$$\phi(z) = ze^{-\imath\pi/4} \quad \Rightarrow \quad \overline{p} := \phi(1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad p := \phi(\imath) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Now consider the function $h(z) = (\phi \circ f \circ \phi^{-1})z$ then $h(z)$ maps the real line to the real line. since $(f \circ \phi^{-1})(\mathbb{R}) = f(L) \subset L$. So for $z = x \in \mathbb{R}$ we have $h(z) = \overline{h(\overline{z})}$.

$$h(p) = (\phi \circ f \circ \phi^{-1})p = (\phi \circ f)1 = \phi(0) = 0$$

and since $h(z) = \overline{h(\overline{z})}$ on the real line we have

$$0 = h(p) = \overline{h(\overline{z})} = (\phi \circ f \circ \phi^{-1})\overline{p} = (\phi \circ f)\imath$$

Which implies that $\phi^{-1}(0) = f(\imath) = 0$ □

Problem: Let $f(z)$ be holomorphic in the upper half plane $U = \{z = x + iy : y > 0\}$ and continuous on $\overline{U}$. Assume $f(x) = ix^3$ for all $x \in (0, 10)$. Find all such $f(z)$.

Solution: Let $g(z) = f(z) - iz^3$, then $g(z)$ is holomorphic on U. Define U_0 as follows:

$$U_0 = \{z = x + iy : 0 < x < 10, y > 0\}$$

Then $g(z)$ is holomorphic on U_0 and continuous on $U_0 \cup (0, 10)$, furthermore

$$\lim_{z \to (0,10)} \Im(g(z)) = 0, \quad \forall z \in U_0$$

Hence by the Schwarz reflection principle for holomorphic functions, we have

$$\widehat{g}(z) = \begin{cases} g(z) & z \in U_0 \\ g(x) = f(x) - ix^3 & z = x \in (0, 10) \\ \overline{g(\overline{z})} & \overline{z} \in U_0 \end{cases}$$

is holomorphic on $U_1 = \{z = x + iy : 0 < x < 10, y \in \mathbb{R}\}$. Now $\{z \in U_1 : \widehat{g}(z) = 0\} \supset (0, 10)$, which has an accumulation point in V. Thus by uniqueness, $\widehat{g} \equiv 0$ on U_1. This implies $g \equiv 0$ on U_0, and hence $U_0 \subset \{z \in U : g(z) = 0\}$, which has an accumulation point on U. Which again implies, by the uniqueness theorem, $g \equiv 0$ on U. Therefore $f(z) = iz^3$ on U □

Remark: Schwarz reflection principle for holomorphic functions: Let V be a connected open set in $\mathbb{C}$ such that $U_{\mathbb{R}} = V \cap(\text{real axis}) = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. Set $U = \{z \in V : \Im(z) > 0\}$. Suppose that $F : U \to \mathbb{C}$ is holomorphic and that

$$\lim_{z \to x} \Im(F(z)) = 0, \quad z \in U$$

for each $x \in U_{\mathbb{R}}$. Define $\widehat{U} = \{z \in \mathbb{C} : \overline{z} \in U\}$. Then there is a holomorphic function G on $U \cup \widehat{U} \cup V_{\mathbb{R}}$ such that $G\big|_U = F$. In particular,

$$G(z) = \begin{cases} F(z) & z \in U \\ \lim_{z \to x} \Re(F(z)) & z \in U, x \in U_{\mathbb{R}} \\ \overline{F(\overline{z})} & z \in \widehat{U} \end{cases}$$

Problem: Let f(z) be analytic and satisfy $|f(z)| \leq 100|z|^{-2}$ in strip $\alpha_1 \leq \Re(z) \leq \alpha_2$. Prove the function

$$h(x) = \int_{-\infty}^{\infty} f(x + \imath y)dy$$

is a constant function of $x \in [\alpha_1, \alpha_2]$.

Proof: Let $x_1, x_2 \in [\alpha_1, \alpha_2]$ such that $x_1 < x_2$, $R > 0$ and consider the following contour Γ.

$$\Gamma = \begin{cases} \gamma_1 := x_2 + \imath t & t \in [-R, R] \\ \gamma_2 := \quad -t & t \in [x_1 + \imath R, x_2 + \imath R] \\ \gamma_3 := x_1 - \imath t & t \in [-R, R] \\ \gamma_4 := \quad t & t \in [x_1 - \imath R, x_2 - \imath R] \end{cases}$$

Now since $f(x)$ is analytic we know that $\int_\Gamma f(z)\ dz = 0$. Now computing γ_1 we have

$$\int_{\gamma_1} f(z)\ dz = \imath \int_{-R}^{R} f(x_2 + \imath t)\ dt \quad \Rightarrow \quad \imath \int_{-\infty}^{\infty} f(x_2+\imath t)\ dt \text{ as } R \to \infty$$

Also for γ_3 and with the change of variable $y = -t$ we have

$$\int_{\gamma_3} f(z)\ dz = \imath \int_{-R}^{R} f(x_1 - \imath t)\ dt \quad \Rightarrow \quad -\imath \int_{-\infty}^{\infty} f(x_1+\imath y)\ dy \text{ as } R \to \infty$$

Now for γ_2 and z lying on the line γ_2 we have

$$\begin{aligned} \left|\int_{\gamma_2} f(z)\ dz\right| &\leq \int_{x_1+\imath R}^{x_2+\imath R} f(t)\ dt \\ &\leq \int_{x_1+\imath R}^{x_2+\imath R} \frac{100}{|z|^2} \\ &= \frac{100}{|z|^2}(x_2 - x_1) \to 0 \text{ as } R \to \infty \end{aligned}$$

For γ_4 and z lying on the line γ_4 we obtain the same result as in γ_2. This implies that

$$0 = \imath \int_{-\infty}^{\infty} f(x_2+\imath t)\ dt - \imath \int_{-\infty}^{\infty} f(x_1+\imath t)\ dt \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x_2+\imath t)\ dt = \int_{-\infty}^{\infty} f(x_1+\imath t)\ dt$$

Now this is for all $x_1, x_2 \in [\alpha_1, \alpha_2]$. Therefore $h(x)$ must be constant $\square$.

Problem: Prove that the product

$$\prod_{k=1}^{\infty} \left(\frac{z^n}{n!} + \exp\left(\frac{z}{2^n}\right)\right)$$

converges uniformly on compact sets to an entire function.

Solution: Denote the above product $p(z)$. Fix $r > 0$ and consider $D(0, r)$. By the triangle inequality we have

$$\sum_{n=1}^{\infty} 1 - \frac{z^n}{n!} + \exp\left(\frac{z}{2^n}\right) \leq \sum_{n=1}^{\infty} \left|1 - e^{z/2^n}\right| + \sum_{n=1}^{\infty} \left|\frac{z^n}{n!}\right|$$

now the last series converges everywhere on $\mathbb{C}$ to e^z. So all the needs to be shown is the first series $\sum_{n=1}^{\infty} |1 - e^{z/2^n}|$ is finite on compact disks. Now there exists $\rho > 0$ such that for all $z \in D(0, \rho)$ we have $|1 - e^z| \leq c_\rho |z|$. This implies

$$\forall z\ s.t.\ \left|\frac{z}{2^n}\right| < \epsilon \quad \Rightarrow |1 - e^{z/2^n}| \leq c_\rho \frac{|z|}{2^n}$$

Hence we have

$$\sum_{n=1}^{\infty} \left|1 - e^{z/2^n}\right| < c_\rho \sum_{n=1}^{\infty} \frac{|z|}{2^n}, \quad z \in \overline{D}(0, \rho - \epsilon)$$

Therefore the product $p(z)$ converges $\square$

Problem: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z+1) = f(z), \quad |f(z)| \leq e^{|z|}, \quad z \in \mathbb{C}$$

Prove that f must be constant.

Solution: Consider $A = \{z \in \mathbb{C} : 0 \leq \Re(z) <\leq 1\}$, and

$$g(z) = \frac{f(z) - f(1/2)}{\cos(\pi z)} \text{ for } \quad z \in A$$

Now at $g(z)$ is bounded on A, since $z = 1/2$ is a removable singularity. Hence by Louville's theorem $g(z) = c$, this implies that

$$f(z) = f(1/2) + c\cos(\pi z) \quad \Rightarrow \quad |f(z)| = |f(1/2) + c\cos(\pi z)| \leq e^{|z|} \Leftrightarrow c = 0$$

Which implies that $f(z) = f(1/2)$, hence $f(z)$ is constant $\square$

Problem: Let $h : \mathbb{C} \to \mathbb{R}$ be harmonic and non-constant. Show that $h(\mathbb{C}) = \mathbb{R}$

Solution: Suppose that $h(\mathbb{C}) \subset [k, \infty)$ then, $h(z) - k \subset [0, \infty)$. Now consider and entire function $f(z) \in O(\mathbb{C})$ st $\Re(f(z)) = h(z)$ then $e^{-f(z)} \in O\mathbb{C}$ and $|e^{-f}| = e^{-h} \leq 1$. so e^{-h} is constant, which is a contradiction, therefore $h(\mathbb{C}) = \mathbb{R}$.

Problem: Let f be holomorphic in $D(0,2)$ and continuous in $\overline{D}(0,2)$. Suppose that $|f(z)| \leq 16$ for $z \in D(0,2)$ and $f(0) = 1$. Prove that f has at most 4 zeros in $D(0,1)$.

Solution: Consider the following:

$$\ln(f(0)) + \sum_{k=1}^{n} \ln\left|\frac{z}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(2e^{\imath\theta})|\, d\theta \leq \ln(16)$$

Now if $|a_k| < 1$ we have

$$\ln\left|\frac{2}{a_k}\right| \geq \ln|2|$$

Hence we have

$$\ln(2) \leq \sum_{k=1}^{n} \ln\left|\frac{2}{a_k}\right| \leq 4\ln(2)$$

Thus $f(z)$ has at most 4 zeros inside $D(0,1)$ $\square$

Problem: Denote $A = \{r < |z| < R\}$, where $0 < r < R < \infty$. TRUE OR FALSE: For every $\epsilon > 0$ there exists a polynomial $p(z)$ such that

$$\sup\left\{\left|p(z) - \frac{1}{z^2}\right|, z \in A\right\} < \epsilon$$

Solution: The statement is true. Let $\rho = \dfrac{R+r}{2}$ and consider the following:

$$\sup_{|z|=\rho} \frac{1}{|z|^2}\left|p(z)z^2 - 1\right| < \epsilon \qquad \forall \epsilon > 0$$

Fix $\epsilon > 0$ now this implies that

$$\sup_{|z|=\rho} \left|p(z)z^2 - 1\right| < \epsilon\rho^2 = \delta$$

Now let $f(z) = z^2 p(z) - 1$, then $f(0) = -1$, since $f(z)$ is a polynomial $f(z) = u(z) + \imath v(z)$, for some harmonic functions $u(z)$ and $v(z)$. Now $|f(z)|^2 = |u(z)|^2 + |v(z)|^2$, so if $|f(z)|^2 \leq \delta$, then $|u(z)|^2 < \delta$; however on $D(0,\rho)$ we have $|u(0)| = 1$, which is a contradiction to the maximum principle, since $|u(0)| = 1$ and $u(z) < \delta$ $\square$

Infinite productions and their applications

Convergence of infinite products

$$\sum_{k=1}^{\infty} |a_k| < \infty \quad \Leftrightarrow \quad \prod_{k=1}^{\infty}(1 + |a_k|) < \infty \quad \Rightarrow \quad \prod_{k=1}^{\infty}(1 + a_k) < \infty$$

Remark: Let $E_0 = 1 - z$, $E_p = E_0 \exp\left(\sum_{k=1}^{p} \frac{z^k}{k!}\right)$

Lemma: Let $\{a_k\}$ such that $a_k \neq 0$ be a sequence with no accumulation point. If p_j are positive integers such that

$$\sum_{k=1}^{\infty} \left(\frac{r}{|a_k|}\right)^{p_j} < \infty \quad \forall r > 0$$

Then the product

$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$$

converges uniformly on compact subsets of $\mathbb{C}$ to an entire function.

Proof:

$$\left|E_{p_k}\frac{z}{a_k} - 1\right| \leq \left|\frac{z}{a_k}\right|^{p_k+1} \leq \left|\frac{r}{a_k}\right|^{p_k}$$

Now $\sum_{k=1}^{\infty}\left|\frac{r}{a_k}\right|^{p_k} < \infty$ hence $\sum_{k=1}^{\infty}\left|E_{p_k}\frac{z}{a_k} - 1\right| < \infty$, thus $\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right)$ converges normally to an entire function. $\square$

Weierstrass Factorization: Let $f(z)$ be an entire function, and suppose $f(z)$ vanishes at 0 of order m. Let $\{a_k\}$ be the non-zero zeros of $f(z)$. Then there exists an entire function $g(z)$ such that

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{k-1}\left(\frac{z}{a_k}\right)$$

Mittag-Leffler Theorem: Let $U \subset \mathbb{C}$ open, and let $\{a_j\}$ be a set of distinct elements with no accumulation point in U. Suppose for all j, V_j is a neighborhood of a_j such that $a_j \notin V_k$ for $k \neq j$ and suppose that m_j is meromorphic on V_j with only pole a_j. Then there exists a meromorphic function $m(z)$ on U such that $m - m_j$ is holomorphic on V_j for all j which has no poles other then those at a_j

Second version: Let $U \subset \mathbb{C}$ open, and let $\{a_j\}$ be a set of distinct elements with no accumulation point in U. Let s_j be a sequence of Laurent polynomials, i.e.

$$s_j = \sum_{n=-p(j)}^{-1} a_n^j (z - a_j)^n$$

Then there exists a meromorphic function on U whose principle part at each a_j is s_j and has no other poles.

Jensen's formula: Let $f(z)$ be holomorphic in a neighborhood of $\overline{D}(0,r)$. Suppose that $f(0) \neq 0$, let $\{a_k\}$ be the zeros according to their multiplicities, then

$$\ln(f(0)) + \sum_{j=1}^{n(r)} \ln\left|\frac{r}{a_j}\right| = \frac{1}{2\pi}\int_0^{\pi} \ln\left|f\left(re^{\imath\theta}\right)\right| \, d\theta$$

Where $n(r)$ is the number of zeros of $f(z)$ counting multiplicities.

Problem: Show that if $f(z)$ is a nonconstant holomorphic function on $D(0,1)$ with $f(0) \neq 0$ and $\{a_k\}$ as the roots then

$$\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$$

Solution: Let $M = \sup\{|f(z)| : z \in D(0,r)\}$, where $0 < r < 1$, by Jensen's formula we have

$$\ln|f(0)| + \sum_{n=k}^{n(r)} \ln\left|\frac{1}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(re^{\imath\theta})\right| \, d\theta \leq \ln(M)$$

where $n(r)$ is the number of roots inside $D(0,1)$, which implies that

$$\sum_{n=k}^{n(r)} \ln\left|\frac{r}{a_k}\right| \leq \ln(M) - \ln(f(0))$$

Letting $r \to 1^-$ we have

$$\sum_{n=k}^{n(r)} \ln\left|\frac{1}{a_k}\right| \leq \ln(M) - \ln(f(0)) < \infty$$

and

$$\sum_{k=1}^{\infty}(1-|a_k|) \leq \sum_{n=k}^{n(r)} \ln\left|\frac{1}{a_k}\right| < \infty$$

Problem: For which real values of ρ and μ does the following product converges normally on $\mathbb{C}$

$$\prod_{n=1}^{\infty}\left(\frac{n^\mu - z}{n^\rho}\right)$$

Solution: Denote $a_n = n^{\mu-\rho} - \frac{z}{n^\rho}$, now we have the following

$$\prod_{n=1}^{\infty} a_n < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}(1-|a_n|) < \infty$$

Fix $r > 0$ then for all z in $D(0,r)$ we have

$$\frac{n^\mu - z}{n^\rho} \to 1 \quad \Leftrightarrow \quad \mu = \rho, \mu\rho > 0$$

Now let $\mu = \rho \in (0,1]$, then we have

$$\sum_{n=1}^{\infty}\left(1 - \left|1 - \frac{z}{n^\rho}\right|\right) \leq \sum_{n=1}^{\infty}\left|\frac{z}{n^\rho}\right| = \infty$$

Therefore if $\mu = \rho$ and $\mu, \rho > 1$, $\displaystyle\prod_{n=1}^{\infty} a_n < \infty$ □

Problem: Let $f(z)$ be an entire functions such that $|f(z)| = 1$ for each $|z| = 1$. Find all such entire functions.

Solution: Consider $D(0,1)$ and first notice that the set of roots of $f(z)$ must be finite. Now let $\{a_k\}$ be the set of roots of $f(z)$ inside $D(0,1)$ and consider the following function.

$$\phi(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z}$$

Now since $\phi(z)$ shares the same roots as $f(z)$ inside $D(0,1)$ we have $\dfrac{f(z)}{\phi(z)}$ is holomorphic inside $D(0,1)$. Furthermore we have $\left|\dfrac{f(z)}{\phi(z)}\right| = 1$ if $|z| = 1$, this implies that $\dfrac{f(z)}{\phi(z)} = 1$ for all $z \in \overline{D}(0,1)$. Hence $f(z) = \omega\phi(z)$ for some $\omega \in \partial D(0,1)$. Now $f(z)$ is entire which implies that $\overline{a}_k = 0$ for all k. Therefore $f(z) = \omega z^n$, for any $n \in \mathbb{N}$ are all such entire functions □

Problem: Set $U = \{z = x + \imath y : x > 0, |y| < x\}$. Suppose that given a sequence of holomorphic functions $f_n : D \to U$, where D is the unit disk. Prove that if $\displaystyle\sum_{n=1}^{\infty} f(0)$ converges then the series $\displaystyle\sum_{n=1}^{\infty} f(z)$ converges normally on D.

Solution: Consider the translation $g_n(z) = e^{\imath\pi/4} f_n(z)$. Now $g_n(z) = u_n(z) + \imath v_n(z)$ where $u_n(z)$ and $v_n(z)$ are harmonic functions. Also

$$\sum_{n=1}^{\infty} g_n(0) < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} f(0) < \infty$$

Now Harnacks principle says that $u_n(z), v_n(z)$, either converge normally to infinity for all $z \in D(0,1)$, or $u_n(z), v_n(z)$ are finite for all $z \in D(0,1)$. Now

$$\sum_{n=1}^{\infty} g_n(0) = \sum_{n=1}^{\infty} u_n(0) + \imath \sum_{n=1}^{\infty} v_n(0) < \infty$$

by assupmtion. Therefore $\sum_{n=1}^{\infty} g_n(z)$ is finite for all $z \in D(0,1)$, and hence $\sum_{n=1}^{\infty} f_n(z)$ converges normally on $D(0,1)$ $\square$.

Problem: Suppose that $f(z)$ is holomorphic in the unit disk D, continuous on $\overline{D}$, and has the following properties:

a) $|f(0)| = a > 1$
b) $|f(z)| > a^3$ for every $z \in \partial D$
c) $f(z)$ does not have zeroes in $\overline{D}(0, 1/a)$

Prove that $f(z)$ must have at least three zeros in D.

Solution: Let $1 > r > \left|\frac{1}{a}\right|$ and consider $D(0,r)$. Denote $n(r)$ as the number of roots inside $D(0,r)$. Now

$$\ln(f(0)) + \sum_{k=1}^{n(r)} \ln\left|\frac{r}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{\imath\theta}\right)\right| \, d\theta$$

where $\{a_k\}$ are the roots of $f(z)$ inside $D(0,r)$. So we have

$$\ln(a) + \sum_{k=1}^{n(r)} \ln\left|\frac{r}{a_k}\right| > \frac{1}{2\pi}\int_0^{2\pi} \, d\theta = \ln(a^3)$$

which implies that

$$\sum_{k=1}^{n(r)} \ln\left|\frac{1}{a_k}\right| > 2\ln(a) \qquad \text{as} \qquad r \to 1^-$$

Hence $n(r) > 2$, therefore $f(z)$ must have at least 3 zeros inside $D(0,1)$ $\square$.

Problem: Suppose that both series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

converge in some neighborhoos of the origin. Assume that for each $n \in \mathbb{N}$ either $b_n = a_n$ or $b_n = 0$. In other words, the series for $g(z)$ is obtained by "removing" some terms from the series for $f(z)$.

a) Is it possible that the radius of convergence for $g(z)$ is strictly larger than the radius of convergence for $f(z)$? Strictly smaller?

Solution: The raduis of convergence can be larger. Suppose there exists $N \in \mathbb{N}$ such that $b_n = 0$ for all $n > N$, then the radius of convergence for $g(z)$ is infinite since $g(z)$ is a polynomial. Otherwise if there does not exist such an N. Consider the seqence $b_{n_j} = b_n$ if $b_n = a_n$ then we have

$$\lim_{n\to\infty} \sup |a_n|^{-1/n} = \lim_{n\to\infty} \sup |b_{n_j}|^{-1/n} = \lim_{n\to\infty} \sup |b_n|^{-1/n}$$

Hence the radii of convergence are the same.

b) Is it possilbe that the domain of holomorphy for $g(z)$ (the largest open connect set where the function $g(z)$ can be extended) is strictly larger that the domain of holomorphy for $f(z)$? Strictly smaller?

Solution: The domain of holomorphy can be larger or smaller. For both cases consider the following function $f(z)$

$$f(z) = \sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$$

The domain of holomorphy for $f(z)$ is $\mathbb{C} - \{1\}$. Now fix N and let $b_n = 0$ for all $n > N$, then $g(z)$ is a polynomial of degree N, and hence is entire. For a smaller domain consider the following $g(z)$

$$g(z) = \sum_{n=1}^{\infty} z^{2^n}$$

where $b_n = 0$ for the non 2^n terms in $f(z)$. Then the domain of holomorphy for $g(z)$ is exactly $D(0,1)$, which is much smaller then $\mathbb{C} - \{1\}$ $\square$

Problem: Let f be analytic in the unit disk $D(0,1)$ and continuous on $\overline{D}(0,1)$. Assume that

$$|f(z)| = |e^z| \quad \forall z \in \partial D(0,1)$$

Find all such f.

Solution: Let α_i be the zeros of $f(z)$ inside $D(0,1)$. Then there exists only a finite number of $\alpha_i \in D(0,1)$, Otherwise $\{\alpha_i\}$ would have an accumulation point, hence $f(z) \equiv 0$. Consider the Blaschke factors, and the Blaschke product defined by:

$$\phi_j(z) = \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \quad \phi(z) = \prod_{j=1}^{n} \phi_j(z)$$

where n is the number of roots includeing multiplicites. Note that $|\phi(z) = 1|$ when $|z| = 1$, now consider the following:

$$g(z) = \frac{f(z)}{\phi(z)e^z} \quad \rightarrow \quad |g(z)| = 1 \text{ for } z \in \partial D(0,1)$$

Now all sigularities of $g(z)$ are removable, so by Riemanns removeable singularity theorem, there is a holomorphic function $\widehat{g}(z)$ with the same properties as $g(z)$. Now by the maximum modulus principle we have $|g(z)| \leq 1$, for all $z \in D(0,1)$. Also, by the minimum modulus principle we have $|g(z)| \geq 1$ for all $z \in D(0,1)$. therefore we have $g(z) = \omega$, where $\omega \in \partial D(0,1)$.

$$\therefore \; f(z) = \omega\phi(z)e^z \quad z \in \overline{D}(0,1) \; \square$$