Analysis qual study guide

James C. Hateley

1. Real Analysis

Exercise 1.1. *Suppose that $f : [0,1] \to (0,1)$ is a non-decreasing function. Prove or disprove that there exists $x \in (0,1)$ such that $f(x) = x$.*

Proof: First notice note that $0 < f(0) \le f(1) < 1$ and so $f^n : (0,1) \to (0,1)$ for all $n \in \mathbb{N}$. Let $x \in (0,1)$, now if $f(x) \le x$, then $(f \circ f)(x) \le f(x)$. Otherwise if $x \le f(x)$ then $f(x) \le (f \circ f)(x)$ since f is a non-decreasing function. $(WLOG)$ for a given x suppose $x \le f(x)$, then $f(x) \le f^n(x) < 1$ for all $n \in \mathbb{N}$. So $\{f^n(x)\}$ is a bounded monotonic sequnce. Therefore by the monotone convergence theorem for real numbers, there exists a $y \in (0,1)$ such that $f^n(x) \to y$ as $n \to \infty$. Now

$$\lim_{n\to\infty} f^n(x) = y \quad \Rightarrow \quad f(\lim_{n\to\infty} f^n(x)) = f(y) = y$$

since $f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = y$

Exercise 1.2. *Suppose that $\mathcal{X}$ is a compact metric space and that $f : \mathcal{X} \to \mathcal{X}$ is an isometry. Prove that $f(\mathcal{X}) = \mathcal{X}$.*

Proof: If $f(x)$ is an isometry and if d is the metric on $\mathcal{X}$ we have $d(x,y) = d(f(x), f(y))$. Now suppose that $f(x)$ is not surjective, then the set $\mathcal{X}\backslash f(\mathcal{X})$ is non empty, so let $x \in \mathcal{X}\backslash f(\mathcal{X})$. We have then $0 < 2\epsilon = dist(x, f(\mathcal{X}))$ for some $\epsilon > 0$. Since $\mathcal{X}$ is compact, there exists a finite open covering $\mathcal{O}$ such that

$$\mathcal{O} = \bigcup_{i=1}^{N} B(x_i, \epsilon),$$

for some x_i and some N, choose N to be as small as possible. Since $x \in B(x_\alpha, \epsilon)$ for some x_α and $2\epsilon = dist(x, f(\mathcal{X}))$ we have $B(x_\alpha, \epsilon) \subset \mathcal{X}\backslash f(\mathcal{X})$. This contradicts the minimality of this N, therefore f is surjective.

Exercise 1.3. *Let f be a real-valued function defined on $[1,\infty)$, satisfying $f(1) = 1$ and $f'(x) = \dfrac{1}{x^2 + f(x)^2}$. Prove that $\lim_{x\to\infty} f(x)$ exists and is less thatn $1 + \dfrac{\pi}{4}$.*

Proof: First notice that $f'(x) \to 0$ as $x \to \infty$ and that $f'(x) > 0$ and so $f(x)$ is always increasing. Since $f(1) = 1$ and $f(x)$ is always increasing we have $f(x) \ge 1$ for all $x \in [1, \infty)$. Now

$$f'(x) = \frac{1}{x^2 + f(x)^2} \le \frac{1}{x^2 + 1},$$

and so

$$\int_1^x f'(t)\, dt \le \int_1^x \frac{1}{t^2+1}\, dt \le \int_1^\infty \frac{1}{t^2+1}\, dt = \tan^{-1}(t)\big|_1^\infty = \frac{\pi}{4}.$$

The second inequality is justified since we integrate a positive function over a larger set. So we have

$$\int_1^x f'(t)\, dt \le \frac{\pi}{4} \quad \Rightarrow \quad f(x) - f(1) \le \frac{\pi}{4} \quad \Rightarrow \quad f(x) \le 1 + \frac{\pi}{4}$$

and the result is shown.

Exercise 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property:*

$$\liminf_{y\to x} \frac{f(x) - f(y)}{x - y} > 0, \quad \forall x \in \mathbb{R}$$

Prove that f is strictly increasing.

Proof: For all $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - y| < \delta$, then we have

$$\frac{f(x) - f(y)}{x - y} \geq \epsilon. \tag{1.1}$$

Now for a given ϵ consider the covering $\mathbb{R}$ by $B(x_\alpha, \eta_\alpha)$, for $x_\alpha \in \mathbb{R}$ and where $0 < \eta_\alpha$ is chosen such that 1.1 is satified. By Lindelöf's covering theres there is a countable subcover of this. Enumerate this subcover as $B(x_i, \eta_i)$. Choose $y_i, z_i \in B(x_i, \eta)$ such that $y_i < x_i < z_i$. Now for each x_i and z_i we have

$$f(x_i) - f(y_i) \geq \epsilon(x_i - y_i) > 0 \text{ and } f(z_i) - f(x_i) \geq \epsilon(z_i - x_i) > 0$$

Hence we have if, by transitivity,

$$x > y \quad \to \quad f(x) > f(y).$$

In other words f strictly increasing.

Exercise 1.5. *If the function f has a continuous derivative on $[0, 1]$, prove that*

$$\int_0^1 f'(x)\, dx = f(1) - f(0).$$

Proof: By definition of the Reimann integrable integral for the interval $[0, 1]$ we have

$$\int_0^1 f'(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f'(x_i)$$

Where $x_i \in [t_{i-1}, t_i]$ and $\{t_i\}_{i=0}^n$ is a a partition of $[0, 1]$. Now by the mean value theorem for derivatives, for each subinterval $[t_{i-1}, t_i]$ of the partition $[0, 1]$, there exists an c_i such that

$$f'(c_i) = \frac{f'(t_i) - f(t_{i-1})}{t_i - t_{i-1}}$$

Let $x_i = c_i$ for each subinterval in the partition. So we have

$$\int_0^1 f'(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f'(c_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}$$

Now $|t_i - t_{i-1}| = 1/n$, hence we have

$$\int_0^1 f'(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i) - f(t_{i-1}) = \lim_{n\to\infty} (f(1) - f(0)) = f(1) - f(0).$$

Exercise 1.6. *If the function $f : [0, \infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x)$ exists, prove that $f(x)$ is uniformly continuous on $[0, \infty)$.*

Proof: Want to show that:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. if } |x - y| < \delta_\epsilon \Rightarrow |f(x) - f(y)| < \epsilon. \tag{1.2}$$

Now since $\lim_{x\to\infty} f(x)$ this means that first the limit is finite, and then for all $\eta > 0$ there exists $N \in \mathbb{N}$ such that $|f(x) - L| < \eta$ for all $x > N$, where L is the limit. Now consider the interval $[0, N]$, since $[0, N]$ is closed and bounded so continuity implies uniform continuity, so given an $\epsilon > 0$ there is a $\delta_1 > 0$ such that 1.2 is satified. Let $x, y \in [N, \infty)$ since $|f(x) - L| < \eta$ and $|f(y) - L| < \eta$ we have by the triangle inequality

$$|f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < 2\eta = \epsilon$$

This implies that for any $\epsilon > 0$, and N large enoughth if $x, y > N$, then $|f(x) - f(y)| < \epsilon$, denote $\delta_2 = \eta$. So if $|x - y| < \delta_2$, then we have $|f(x) - f(y)| < \epsilon$. In otherwords $f(x)$ is uniformly continuous on (N, ∞). Let $\delta_\epsilon = \min\{\delta_1, \delta_2\}$. This δ_ϵ either depends on ϵ or N, in the later case N depends on ϵ. This implies 1.2 is will be satified, so the result is shown.

Exercise 1.7. *If $A \subset \mathbb{R}^n$ is nonempty, define the distance d_A of $x \in \mathbb{R}^n$ to A by*

$$d_A(x) = \inf\{\|x - z\| : z \in A\}$$

(a) Show that if $x, y \in mR^n$ then $|d_A(x) - d_A(y)| \leq \|x - y\|$.
(b) If $n = 2$ and $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\}$, find a point in $\mathbb{R}^2$ at which d_A is not differentiable. Justify your claim.

Proof: For part (a) we can think of z as defined in the function d_A like an orthogonal projection to A, then d_A returns the length of this projection. Let $x, y \in \mathbb{R}^n$ and $z \in A$

$$\begin{aligned} d_A(x) \leq \|x - z\| \leq \|x - y\| + \|y - z\| \quad &\Rightarrow \quad d_A(x) - \|y - z\| \leq \|x - y\| \\ d_A(y) \leq \|y - z\| \leq \|y - x\| + \|x - z\| \quad &\Rightarrow \quad d_A(y) - \|x - z\| \leq \|x - y\| \end{aligned}$$

Now subtracting we can obtain the following:

$$\begin{aligned} d_A(x) - d_A(y) \leq \|y - z\| - \|z - x\| \leq \|y - x\| \\ d_A(y) - d_A(x) \leq \|x - z\| - \|z - y\| \leq \|y - x\| \end{aligned}$$

which implies $|d_A(x) - d_A(y)| \leq \|x - y\|$.

(b) Consider the point $(0, 1)$. Then we have $d_A(0, 1) = 0$. Now

$$\lim_{h \to 0} \frac{d_A(0, 1 + h) - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and

$$\lim_{h \to 0^+} \frac{d_A(h, 1) - 0}{h} = \lim_{h \to 0^+} \frac{\sqrt{1 + h^2}}{h} = \infty$$

The one-sided partial ∂_{x^+} is unbounded at $(0, 1)$ hence the derivative cannot exist.

Exercise 1.8. *Let $\{f_n\}$ be a sequence of functions on $[0, 1]$ such that*

$$\sup_{x \in [0,1]} |f_n(x)| = M_n < \infty, \quad n \in \mathbb{N}.$$

Suppose $\{f_n\}$ converges uniformly on $[0, 1]$. Prove that there exists M, $0 \leq M < \infty$, such that for all $x \in [0, 1]$ and $n \in \mathbb{N}$ the inequality $|f_n(x)| \leq M$ holds.

Proof: If $\{f_n(x)\}$ converges uniformly, then there exists a function f such that for each $x \in [0, 1]$ and for all $1 > \epsilon > 0$ there is an N such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N$. So

$$|f(x)| = |f(x) - f_{N+1}(x) + f_{N+1}(x)| \leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x)| < \epsilon + |f_{N+1}(x)| < \epsilon + M_{N+1}$$

For this x, let $M = \max_{i=1..N}\{M_i, 1 + M_{N+1}\}$, then we have $f_n(x) \leq M$ for all $n \in N$. Now if $n > N$ then we have for all $x \in [0, 1]$, $|f_n(x)| < \epsilon + M_{N+1}$, since $|f_n(x) - f(x)| < \epsilon$. If $n \leq N$ then $f_n(x) < M_n < M$. So the result holds for all $x \in [0, 1] and n \in \mathbb{N}$.

Exercise 1.9. *Let $\sum a_n, \sum b_n$ be two series with positive terms. Assume that $\sum b_n$ converges and that*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}, \quad \forall n \in \mathbb{N}$$

Prove that $\sum a_n$ converges.

Proof: Consider the two power series

$$A = \sum_{n=1}^{\infty} a_n x^n, \quad B = \sum_{n=1}^{\infty} b_n x^n$$

Series B will converge if

$$\frac{b_{n+1}}{b_n}|x| < 1 \quad \Rightarrow \quad |x| < \frac{b_n}{b_{n+1}}$$

Now we know for $x = 1$ power series B converges, so

$$1 < \frac{b_n}{b_{n+1}} \text{ or } \frac{b_{n+1}}{b_n} < 1.$$

This implies that

$$\frac{a_{n+1}}{a_n}|x| \le \frac{b_{n+1}}{b_n}|x| < 1 \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} < 1$$

Or power series A will converge for $|x| = 1$ and the result is shown.

Exercise 1.10. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at every point $(x, y) \neq (0, 0)$, and there exists $\alpha \in \mathbb{R}$ so that*

$$f(tx, ty) = t^\alpha f(x, y), \quad \forall t > 0, (x, y) \in \mathbb{R}^2$$

Show that for any $(x, y) \neq (0, 0)$

$$x f_x(x, y) + y f_y(x, y) = \alpha f(x, y)$$

Proof: Taking derivative with respect to t we have

$$x f_x(tx, ty) + y f_y(tx, ty) = \alpha t^{\alpha-1} f(x, y)$$

This equation holds for $t > 0$ and $(x, y) \in \mathbb{R}^2$. In particular, let $t = 1$ and the result is shown.

Exercise 1.11. *Suppose that a function $f \ge 0$ and that*

$$\int_4^\infty f(x)\ dx = 5.$$

Show that

$$\int_2^\infty f(x^2)\ dx \le \frac{5}{4}$$

Proof: First consider the change of variable $u = x^2$, then the differentials are given by $du = 2xdx$ since $x > 0$, we have $x = \sqrt{u}$ and so $dx = \frac{du}{2\sqrt{u}}$. With this change of variable the integral

$$\int_2^\infty f(x^2)\ dx = \frac{1}{2}\int_4^\infty \frac{f(u)}{\sqrt{u}}\ du \le \frac{1}{4}\int_4^\infty f(u)\ du = \frac{5}{4}$$

Since $\dfrac{1}{\sqrt{u}} \le \dfrac{1}{2}$ for $u \in [4, \infty)$

Exercise 1.12. *For positive numbers a, b, show that $\lim_{n\to\infty}(a^n + b^n)^{1/n} = \max\{a, b\}$.*

Proof: Suppose that $0 < a < b$. Let $y = (a^n + b^n)^{1/n}$, taking log and the limit as n goes to infinity we have

$$\begin{aligned}
\lim_{n\to\infty} \ln(y) &= \lim_{n\to\infty} \frac{\ln(a^n + b^n)}{n} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{a^n \ln(a) + b^n \ln(b)}{a^n + b^n} \\
&= \lim_{n\to\infty} \frac{\left(\frac{a}{b}\right)^n \ln(a) + \ln(b)}{\left(\frac{a}{b}\right)^n + 1} \\
&= \ln(b)
\end{aligned}$$

since $\frac{a}{b} \to 0$. The function $\ln(y)$ is continuous for $y > 0$ and so $\lim_{n\to\infty} \ln(y) = \ln(b)$ implies that $\ln(\lim_{n\to\infty} y) = \ln(b)$, which implies $\lim_{n\to\infty} y = b$.

Exercise 1.13. *Let $f : [-1, 1] \to \mathbb{R}$*

(a) *Give a clear definition of f is Reimann integrable on $[-1, 1]$.*
(b) *Using your definition show that the function*

$$g(x) = \begin{cases} 0 & -1 \le x < 0 \\ 5 & x = 0 \\ x & 0 < x \le 1 \end{cases}$$

is Riemann integrable and find $\int_{-1}^1 g(x)\ dx$.

Proof: For part (a), a function is said to be Riemann integrable over an interval $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if P is a partition of $[a, b]$ with $\text{mesh}(P) < \delta$, then

$$\sum_{i=1}^{n}\left(\sup_{x\in[t_{i-1},t_i]}\{f(x)\} - \inf_{x\in[t_{i-1},t_i]}\{f(x)\}\right)(t_i - t_{i-1}) < \epsilon \tag{1.3}$$

where $t_0 = a$ and $t_n = b$.

For part (b), let P be a partition $\{t_i\}$, and $\text{mesh}(P) < 1/n$, also let $0 \in (t_k, t_{k+1}$, that is zero is not an endpoint of this partition. Now by definition we have

$$\begin{aligned}
& \sum_{i=1}^{n}\left(\sup_{x\in[t_{i-1},t_i]}\{f(x)\} - \inf_{x\in[t_{i-1},t_i]}\{f(x)\}\right)(t_i - t_{i-1}) \\
= & \sum_{i=k+1}^{n}\left(\sup_{x\in[t_{i-1},t_i]}\{f(x)\} - \inf_{x\in[t_{i-1},t_i]}\{f(x)\}\right)(t_i - t_{i-1}) \\
\le & \frac{5}{n} + \sum_{i=k+2}^{n}(t_i - t_{i-1})^2 \\
= & \frac{5}{n} + \sum_{i=k+1}^{n}\frac{1}{n}^2 < \frac{6}{n} \to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence the function is Riemann integrable. Note that for $i < k$ the functional value is zero, so this will not contribute towards the sum and $\left(\sup_{x\in[t_k,t_{k+1}]}\{f(x)\} - \inf_{x\in[t_k,t_{k+1}]}\{f(x)\}\right)(t_i - t_{i-1}) < 5/n$ since $\text{mesh}(P) < 1/n$, $\sup_{x\in[t_k,t_{k+1}]}\{f(x)\} = 5$ and $\inf_{x\in[t_k,t_{k+1}]}\{f(x)\} = 0$. Now to find the value of this integral we have

$$\int_{-1}^{1} g(x)\ dx = \int_{0}^{1} x\ dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

since $x = 5$ is a set of measure zero and sets of measure will not contribute towards the value of the integral.

Exercise 1.14. *If the terms of a series*

$$A = \sum_{n=1}^{\infty} a_n$$

are nonnegative and decrease monotonically to zero, then show that the series converges if and only if the following related series converges

$$B = \sum_{j=1}^{\infty} 2^j a_{2^j}$$

Proof: This is Cauchy condensation test. ($\Rightarrow$) Suppose the series A converges. Then for n large enough we must have $a_n < 1/n$, so let n be large enough, since A converges we know by the root test we have

$$\sqrt[n]{|a_n|} < 1 \quad \Rightarrow \quad \sqrt[j]{|2^j a_{2^j}|} = 2\sqrt[j]{|a_{2^j}|} < 2\frac{1}{\sqrt[j]{2^j}} = 1$$

hence the series B converges by the root test.

($\Leftarrow$) Now suppose the series B converges, grouping the series A we have in terms of powers of 2

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= a_1 + \{a_2 + a_3\} + \{a_4 + a_5 + a_6 + a_7\} + \cdots \\
&\le a_1 + \{a_2 + a_2\} + \{a_4 + a_4 + a_4 + a_4\} + \cdots \\
&= a_1 + \sum_{j=1}^{\infty} 2^j a_{2^j} < \infty
\end{aligned}$$

Hence series A converges.

Exercise 1.15. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy*

$$0 \leq f(x,y) \leq |(x,y)|^{1.02}$$

for all (x,y). Find $f'(x,y)$ or show by example that it need not exist.

Proof: $f'(x,y)$ need not exist. Consider the function

$$f(x,y) = \begin{cases} |(x,y)|^{1.02} & \|(x,y)\| \geq 1 \\ 0 & \|(x,y)\| < 1 \end{cases}$$

It is clear the inequality in the problem holds for all $(x,y) \in \mathbb{R}^2$. The function $f(x,y)$ is not continuous on $\|(x,y)\| = 1$, hence the derivative does not exist on $\mathbb{R}$.

Exercise 1.16. *Let f be a differentiable real function defined on $(0,1)$. Show that f' maps $(0,1)$ onto an interval.*

Proof: I shall show a slightly more generalized version of this question. That is, if $f(x)$ is a differentiable function defined on an open interval I then $f'(I)$ can be written as an interval. First if $f(x)$ is a linear function or a constant, then $f'(x)$ is constant so the result holds. Now suppose $f(x)$ is neither of these. Let $x_1, x_2 \in f(I)$ with $x_1 < x_2$ and let $y \in (x_1, x_2)$, we need to show that $y \in f'(I)$. Let $a, b \in I$ such that $f'(a) = x_1$ and $f'(b) = x_2$. Consider the function $g(x) = f(x) - yx$, then $g'(x) = f'(x) - y$. Now the function $g(x)$ is a continuous function, hence the restriction of g to the interval $[a,b]$ is continuous. Now recall that the continuous image of a compace space is compact. Hence g attains a minimum on the interval $[a,b]$. By construction we have $g'(a) < 0$ and $g'(b) > 0$. It is clear that the minimum cannot occur at a or b since $g'(a), g'(b) \neq 0$, hence there is a $c \in (a,b)$ such that $g'(c) = 0$ this implies that $f'(x) = y$. In other words $f'(I)$ is an interval.

2. Complex Analysis

Exercise 2.1. *Prove or disprove that*

$$\int_0^{2\pi} e^{(e^{\imath\theta})}\, d\theta = 0.$$

Proof: This is not true, to see why let us compute the integral. Writing the power series for e^z with $z = e^{\imath\theta}$ and integrating we have

$$\begin{aligned} \int_0^{2\pi} e^{(e^{\imath\theta})}\, d\theta &= \int_0^{2\pi} \sum_{k=0}^{\infty} \frac{e^{k\imath\theta}}{k!}\, d\theta \\ &= \sum_{k=0}^{\infty} \int_0^{2\pi} \frac{e^{k\imath\theta}}{k!}\, d\theta \\ &= \int_0^{2\pi} d\theta + \sum_{k=1}^{\infty} \int_0^{2\pi} \frac{e^{k\imath\theta}}{k!}\, d\theta \\ &= 2\pi - \imath \sum_{k=1}^{\infty} \frac{e^{k\imath\theta}}{(k-1)!}\Big|_0^{\pi} \\ &= 2\pi \end{aligned}$$

We can switch the integral and sum since the sum converges uniformly. Also the value of $\frac{e^{\imath\theta}}{(k-1)!}\Big|_0^{\pi}$ is computed to be zero using Euler's formula.

Exercise 2.2. *Let a and b be distinct complex numbers that lie in the interior of the left half-plane. Prove that $|e^a - e^b| < |a - b|$*

Proof: First notice that if $a = x + \imath y$, then we have

$$|e^a| = |e^{x+\imath y}| = |e^x| < 1$$

Now let $a, b \in \mathbb{C}$ such that $\Re(a), \Re(b) < 0$. By the mean value theorem, there exists a c such that c lies on the line $r(t) = a(1-t) + tb, t \in [0,1]$ and

$$e^c = \frac{e^a - e^b}{a - b}$$

now the line $r(t)$ connecting a and b is contained in the left half-plane hence for all $t \in [0,1], \Re(r(t)) < 0$ and hence we have

$$\left|\frac{e^a - e^b}{a - b}\right| = |e^c| < 1 \quad \Rightarrow \quad |e^a - e^b| < |a - b|$$

Exercise 2.3.

(a) Find the radius of convergence of the Taylor series for $f(z)$ at $z = 1$.

$$f(z) = \frac{1}{z^4 + z^2 + 1}$$

(b) Explicity find the constant term and the linear term of the series.

Proof: The Taylor series about $z = 1$ is given by

$$f(z) = \sum_{k=1}^{\infty} a_k (z-1)^k, \qquad a_k = \frac{f^{(k)}(1)}{k!}.$$

Instead of computing the coeffients a_k, the series must avoid the sigularities the radius will be the distance from the center to the closest singularity, i.e. $r = \inf\{|z_k - 1|\}$, where z_k are the roots of $f(z)$. To compute the roots let $w = z^2$,

$$w^2 + w + 1 \quad \Rightarrow \quad w = \frac{-1 \pm \sqrt{3}\imath}{2} = e^{2\pi\imath/3}, e^{4\pi\imath/3}$$

Hence we have

$$z = \{\pm e^{\pi\imath/3}, \pm e^{2\pi\imath/3}\}, \quad z_k = \{e^{k\pi\imath/3} : k = 1,2,4,5\}$$

Considering the complex plane we have

$$r = \inf\{|z_k - 1|\} = \|1 - e^{k\pi\imath/3}\| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

Now using the definition of the Taylor series we have that

$$a_0 = f(1) = \frac{1}{3}, \quad a_1 = f'(1) = \frac{-4z^3 - 2z}{(z^4 + z^2 + 1)^2}\Big|_{z=1} = -\frac{2}{3}.$$

Exercise 2.4. *Compute the following integral using residues. You may leave your solution as a sum of explicity complex numbers.*

$$\int_{\mathbb{R}} \frac{x^4 + 1}{x^6 + 1}\, dx$$

Proof: This is an application of the Residue Theorem. The roots of $z^4 + 1$ are computed to be $\{e^{k\pi\imath/4} : k = 1,3,5,7\}$ and the roots of $z^6 + 1$ are computed to be $\{e^{k\pi\imath/6} : k = 1,3,5,7,9,11\}$. Now consider the following contour:

$$\Gamma = \begin{cases} t & t \in [-R, R] \\ Re^{\imath t} & t \in [0, \pi] \end{cases}$$

By the Residue Theorem we have

$$\int_\Gamma \frac{z^4 + 1}{z^6 + 1}\, dz = 2\pi \sum_k \text{Res}(f(z_k)), \quad z_k \in \Omega, \text{ where } \partial\Omega = \Gamma.$$

Now first let's compute the integral by evalutating on the contour.

$$\int_\Gamma \frac{z^4+1}{z^6+1}\,dz = \int_{-R}^{R} \frac{t^4+1}{t^6+1}\,dt + \int_0^\pi \imath R\frac{R^4e^{4it}+1}{R^6e^{6\imath t}+1}\,dt$$

For the first integral, if $\lim_{R\to\infty}$ this converges to the original integral in question. For the second integral we have

$$\begin{aligned}\left|\int_0^\pi \imath R\frac{R^4e^{4it}+1}{R^6e^{6\imath t}+1}\,dt\right| &\leq \int_0^\pi \left|\imath R\frac{R^4e^{4it}+1}{R^6e^{6\imath t}+1}\right|\,dt\\ &\leq R\frac{|R^4e^{4it}+1|}{R^6-1}\,dt\\ &\leq \frac{2R^5}{R^6-1}\,dt \to 0 \text{ as } R\to\infty\end{aligned}$$

Hence we have

$$\int_{\mathbb{R}} \frac{t^4+1}{t^6+1}\,dt = 2\pi\imath\sum_k \operatorname{Res}(f(z_k)).$$

Now the roots that lie inside Γ are $\{e^{k\pi\imath/6} : k=1,3,5\}$ and so we have

$$\int_{\mathbb{R}} \frac{t^4+1}{t^6+1}\,dt = 2\pi\imath\sum_{j=1}^{3}\left(e^{2(2j-1)\pi\imath/3}-1\right)\left[\prod_{k=1,j\neq k}^{6}\left(e^{(2j-1)\pi\imath/6}-e^{(2k-1)\pi\imath/6}\right)^{-1}\right]$$

Exercise 2.5. *Suppose that f is a non-constant entire function. Prove or disprove each of the following statements.*

(a) The range of f is dense in $\mathbb{C}$.
(b) The range of f is all of $\mathbb{C}$.

Proof: For statement (a), suppose $f(z)$ is a non-constant entire function that is not dence in all of $\mathbb{C}$, then there exists a $w\in\mathbb{C}$ and an $r>0$ such that the $B(w,r)\not\subset f(\mathbb{C})$. Define the function

$$g(z) = \frac{1}{f(z)-w}.$$

Since $f(z)$ is entire, it is clear that $g(z)$ is entire and we have the bound

$$|g(z) \;=\; \frac{1}{|f(z)-w|} < \frac{1}{r}.$$

By Liouville's theorem $g(z)$ is a bounded entire function thus must be constant, hence $f(z)$ is constant. This is a contradiction, hence the range of $f(z)$ must be dense in $\mathbb{C}$.

For part (b) this statment need not be true, consider the function $f(z)=e^z$. This is an entire function, but this function never attains the value zero, hence the image of $f(z)$ is not all of $\mathbb{C}$.

Theorem: (Little Picard) If a function $f:\mathbb{C}\to\mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Theorem: (Big Picard) If an analytic function $f(z)$ has an essential singularity at a point w, then on any punctured neighborhood of w, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

Exercise 2.6. *Let $f(z)$ be an analytic function on $B_r(z_0)\backslash\{z_0\}$, the punctured disk of radius r. Show that if f is bounded, then f can be analytically extended to the whole disk. That is, there is an analytic function $F(z)$ on $B_r(z_0)$ whose restriction on $B_r(z_0)\backslash\{z_0\}$ is $f(z)$.*

Proof: Suppose $f(z)$ is bounded on the $B_r(z_0)\backslash\{z_0\}$ punctured disk. Let

$$M = \sup_{z\in B_r(z_0)\backslash\{z_0\}}\{f(z)\},$$

this is finite by hypothesis. Let $z \neq z_0$ be in $B_r(z_0)$, and let $0 < \epsilon < |r - |z_0||$. Now

$$\begin{aligned}\left|\int_{|w-z_0|=r} \frac{u(w)}{w-z}\ dw\right| &= \left|\int_{|w-z_0|=\epsilon} \frac{u(w)}{w-z}\ dw\right| \\ &\leq \int_{|w-z_0|=\epsilon} \left|\frac{u(w)}{w-z}\right|\ dw \\ &\leq 2\pi\frac{M}{r}\epsilon \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

So fix $0 < R < |r - |z_0||$, then define

$$\tilde{u}(z) = \frac{1}{2\pi\imath}\int_{|w-z_0|=R} \frac{u(w)}{w-z}\ dw$$

for $|z - z_0| < R$. By the Residue theorem we have $\tilde{u}(z) = u(z)$ and furthermore $\tilde{u}(z)$ is analytic for $|z - z_0| < R$, so the result is shown.

Exercise 2.7. *Prove that if $u(x, y)$ is harmonic in a simply connected domain Ω then the line integral*

$$\int_{z_0}^{z} (-u_y\ dx + u_x\ dy)$$

is independent of the path in Ω from z_0 to z.

Proof: Using Green's Theorem we have

$$\int_{\partial\Omega} (-u_y\ dx + u_x\ dy) = \int_{\Omega} u_{xx} + u_{yy}\ dA = \int_{\Omega} \delta u\ dA = 0$$

since $u(x, y)$ is harmonic. Now let $\gamma_1 : z_0 \to z$ be any path and choose $\gamma_2 : z \to z_0$ such that the region enclosed is contained inside Ω and forms a simply connected domain. We then have

$$0 = \int_{\gamma_1\cup\gamma_2} (-u_y\ dx + u_x\ dy) \quad = \quad \int_{\gamma_1} (-u_y\ dx + u_x\ dy) + \int_{\gamma_2} (-u_y\ dx + u_x\ dy)$$

This means we have

$$\int_{\gamma_1} (-u_y\ dx + u_x\ dy) = -\int_{\gamma_2} (-u_y\ dx + u_x\ dy) = \int_{-\gamma_2} (-u_y\ dx + u_x\ dy)$$

This holds for all such paths, hence the integral is independant of path from z_0 to z.

Exercise 2.8. *The Bessel function $J_0(z)$ is defined by*

$$J_0(z) = \frac{1}{2\pi\imath}\int_{\Gamma} \exp(z(w - w^{-1})/2)\frac{dw}{w},$$

where Γ is any simple closed curve around the origin counterclockwise. Compute the Taylor series expansion for $J_0(z)$ by carrying out the following steps:

(a) Expand $\exp(zw/2)$ and $\exp(-z/(2w))$ as a power series.
(b) Multiply them formally to get a Laurent expansion, in the variable w, for $\exp(z(w - w^{-1})/2)$.
(c) Switch the sums and integral.
(d) Apply the Residue Theorem. Don't forget the extra $1/w$ in the integral

Proof: First the Talor expanions for $\exp(zw/2)$ and $\exp(-z/(2w))$ are given as:

$$\exp(zw/2) = \sum_{k=0}^{\infty} \frac{(zw)^k}{2^k k!} \qquad \exp(-z/(2w)) = \sum_{k=0}^{\infty} \frac{(-z)^k}{(2w)^k k!}$$

applying the Cauchy product of these two we have

$$\begin{aligned}\exp(zw/2)\exp(-z/(2w)) &= \left(\sum_{k=0}^{\infty}\frac{zw^k}{2^k k!}\right)\left(\sum_{k=0}^{\infty}\frac{(-z)^k}{(2w)^k k!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-z)^k}{(2w)^k k!}\frac{(zw)^{n-k}}{2^{n-k}(n-k)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k z^k z^{n-k}}{2^k 2^{n-k} n!(n-k)!}\frac{w^n}{w^{2k}}\end{aligned}$$

Hence we have

$$J_0(z) = \frac{1}{2\pi\imath}\int_{\Gamma}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^k z^k z^{n-k}}{2^k 2^{n-k} n!(n-k)!}\frac{w^n}{w^{2k+1}}\ dw.$$

Since the sum converges uniformly to the function $\exp(z(w-w^{-1})/2)$ we can interchange integration and summation. Using the Residue Theorem we have:

$$\begin{aligned}J_0(z) = \frac{1}{2\pi\imath}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k z^k z^{n-k}}{2^k 2^{n-k} n!(n-k)!}\int_{\Gamma}\frac{w^n}{w^{2k+1}}\ dw &= \frac{1}{2\pi\imath}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k z^k z^{n-k}}{2^k 2^{n-k} n!(n-k)!}(2\pi\imath)_{n=2k}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^k z^{2k}}{2^{2k}(n!)^2}.\end{aligned}$$

Since the integral

$$\int_{\Gamma}\frac{w^n}{w^{2k+1}}\ dw = 0, \quad n \neq 2k$$

Exercise 2.9. *Suppose $w = f(z)$ is entire and not constant. Show that the image set $f(\mathbb{C})$ intersects every open set in the w-plane.*

Proof: Let $f(z)$ be an entire and not-constant function ans suppose that there is an open set such that $f(z)$ does not intersect all open sets in the w-plane. Then there is a $z_0 \in \mathbb{C}$ and an $r > 0$ such that $f(\mathbb{C})$ intersect $B_r(z_0)$ is empty. Consider the function $g(z) = (f(z) - z_0)^{-1}$. The function $g(z)$ is entire and we have

$$|g(z)| = \frac{1}{|f(z) - z_0|} < \frac{1}{r}, \quad \forall z \in \mathbb{C}.$$

The function $g(z)$ is an entire bounded function hence by Lioville's Theomre is it constant, hence $f(z)$ is is consant. Therefore the the image set of $f(\mathbb{C})$ intersects every open set in $\mathbb{C}$.

Theorem: (Residue:) Let Γ be a simple closed curve with and let a point inside gamma. Suppose $f(z)$ is a function with no singularities inside Γ and let n be a positive integer, then we have

$$\int_{\Gamma}\frac{f(z)}{(z-a)^n} = 2\pi\imath\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}f(a)$$

Exercise 2.10. *Evaluate the integral*

$$\frac{1}{2\pi\imath}\int_{C}\frac{z\exp(z)}{(z-a)^3}\ dz$$

Assuming the point a lies inside the simple closed curve C.

Proof: This is an application of the Residue Theorem. Applying the Residue theorem we have

$$\frac{1}{2\pi\imath}\int_{C}\frac{z\exp(z)}{(z-a)^3}\ dz = \frac{1}{2}e^a(2+a)$$

Exercise 2.11. *Compute the definite integral*

$$\int_0^{\infty}\frac{1-\cos(x)}{x^2}\ dx$$

Proof: Consider the following identity and change of variables

$$\frac{1-\cos(2x/2)}{x^2} = \frac{2\sin^2(x/2)}{x^2} \qquad y = x/2.$$

Then we have

$$\begin{aligned}\int_0^\infty \frac{1-\cos(x)}{x^2}\,dx &= 2\int_0^\infty \frac{\sin(x/2)}{x^2}\,dx \\ &= 2\int_0^\infty \frac{\sin^2(y)}{(2y)^2}\,2dy \\ &= \int_0^\infty \frac{\sin^2(y)}{y^2}\,dy = \frac{\pi}{2}\end{aligned}$$

This is integrated by considering the following contour:

$$\Gamma = \begin{cases}\gamma_1 := \quad t & t \in [-R, -1/R] \\ \gamma_2 := e^{\imath t}/R & t \in [\pi, 2\pi] \\ \gamma_3 := \quad t & t \in [1/R, R] \\ \gamma_4 := Re^{\imath t} & t \in [0, \pi]\end{cases}$$

Now our function has a removable singularity at $x = 0$, so consider the following

$$f(x) = \frac{1-e^{2\imath x}}{2x^2} \qquad \Rightarrow \qquad \Re(f(x)) = \frac{1-\cos(2x)}{2x^2} = \frac{\sin^2(x)}{x^2}$$

Now for the integral around Γ we have

$$\int_\Gamma f(z)\,dz = 2\pi\imath\,\mathrm{Res}(f(z)) = 2\pi\imath \lim_{z\to 0}\frac{d}{dz}z^2 f(z) = 2\pi\imath \lim_{z\to 0} -\imath e^{2\imath z} = 2\pi\imath(-\imath) = 2\pi$$

Now for the integral on γ_1 we have

$$\int_{\gamma_1} f(z)\,dz = \frac{1}{2}\int_{-R}^{-1/R}\frac{1-e^{2\imath t}}{t^2}\,dt \qquad \Rightarrow \qquad \frac{1}{2}\int_{-\infty}^{0}\frac{1-e^{2\imath t}}{t^2}\,dt \text{ as } R\to\infty$$

For the integral on γ_2 we have

$$\begin{aligned}\int_{\gamma_2} f(z)\,dz &= \frac{1}{2}\int_\pi^{2\pi}\frac{\left(1-e^{2\imath e^{\imath t}/R}\right)\imath e^{\imath t}/R}{e^{2\imath t}/R^2} \\ &= \frac{\imath}{2}\int_\pi^{2\pi}\frac{1-e^{2\imath e^{\imath t}/R}}{e^{2\imath t}/R}\end{aligned}$$

Now letting $R\to\infty$ and using L'Hospitals rule we have

$$\frac{\imath}{2}\int_\pi^{2\pi}\frac{-e^{2\imath e^{\imath t}/R}\,(2\imath e^{\imath t}/R)\,(\imath)}{\imath e^{\imath t}/R} = \int_\pi^{2\pi}\,dt = \pi$$

For the integral on γ_3 we have

$$\int_{\gamma_3} f(z)\,dz = \frac{1}{2}\int_{1/R}^{R}\frac{1-e^{2\imath t}}{t^2}\,dt \qquad \Rightarrow \qquad \frac{1}{2}\int_0^\infty\frac{1-e^{2\imath t}}{t^2}\,dt \text{ as } R\to\infty$$

Now for γ_4 we have

$$\begin{aligned}\int_{\gamma_4} f(z)\,dz &= \frac{1}{2}\int_0^\pi \frac{1-Re^{2\imath t}}{R^2e^{2\imath t}}R\imath e^{\imath t}\,dt \\ &= \frac{\imath}{2}\int_0^\pi\frac{1-Re^{2\imath t}}{Re^{\imath t}}\end{aligned}$$

Putting this all together we have

$$2\pi = \pi + \frac{1}{2}\int_{-\infty}^0\frac{1-e^{2\imath t}}{t^2}\,dt + \frac{1}{2}\int_0^\infty\frac{1-e^{2\imath t}}{t^2}\,dt$$

Taking real parts we have

$$\pi = \int_{-\infty}^{0} \frac{\sin^2(t)}{t^2}\ dt + \int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt = 2\int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt$$

Hence we have $\displaystyle\int_{0}^{\infty} \frac{\sin^2(t)}{t^2}\ dt = \frac{\pi}{2}$ $\square$

Exercise 2.12. *Let f and g be two entire functions such that for all $z \in \mathbb{C}$, $\Re(f(z)) \leq \Re(g(z))$. Prove or disprove that there is a complex number z_1 such that for all $z \in \mathbb{C}$, $f(z) = g(z) = z_1$.*

Proof: text

Exercise 2.13. *Define a one-to-one conformal map from the semidisc*

$$D = \left\{ z \in \mathbb{C} : \Re(z) > 0, \left| z - \frac{\imath}{2} \right| < \frac{1}{4} \right\}$$

onto the upper-half plane $H = \{z \in \mathbb{C} : \Im(z) > 0\}$. Explain, by citing theorems or giving details, why your map is conformal and why it is 1-1 from D onto H.

Proof: First consider sequence of transformations

$$\phi_1 : z \to z - \frac{\imath}{2}, \quad \phi_2 : z \to 4z, \quad \phi_3 : z \to z^2, \quad \phi_4 : z \to \imath\left(\frac{1+z}{1-z}\right)$$

The first transformation $\phi_1(z)$ is a translation down by $1/2$ so that the center of the semidisk is at the origin. The second $\phi_2(z)$ is a dilation to the right unit semidisk. The third transformation $\phi_3(z)$ is maps the semidisk to the unit disk. The last transformation $\phi_4(z)$ is the inverse Cayley transform that maps the unit disk into the upper-half plane. The composition that will take D to H is as follows

$$\phi(z) = (\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1)(z)$$

Exercise 2.14. *Let u and v be real-valued functions defined in a connected open set D of $\mathbb{C}$ such that $f = u + \imath v$ is holomorphic in D. Suppose there are real constants α, β and γ such that $\alpha^2 + \beta^2 \neq 0$ and $\alpha u + \beta v = \gamma$ in D. Prove that f is constant in D.*

Proof: Consider the Cauchy Riemann equations with the condition that $\alpha u + \beta v = \gamma$.

$$C.R. = \begin{cases} u_x & = v_y \\ u_y = -v_x & \end{cases}$$

Taking partial with respect to x and y and using the we have

$$\begin{cases} \alpha u_x + \beta v_x = 0 \\ \alpha u_y + \beta v_y = 0 \end{cases} \Rightarrow \begin{cases} \alpha u_x - \beta u_y = 0 \\ \alpha u_y + \beta u_x = 0 \end{cases} \Rightarrow u_x = \frac{\beta}{\alpha} u_y$$

$\alpha \neq 0$ since $\alpha^2 + \beta^2 \neq 0$. This implies that

$$\alpha u_y + \beta\left(\frac{\beta}{\alpha}\right)) u_y \quad \Rightarrow \quad \left(\alpha^2 + \beta^2\right) u_y = 0$$

which implies that $u_y = v_x = 0$, hence u and v are constant, hence $f(z) = u + \imath v$ is constant in D.

Exercise 2.15. *Let E be the union of two coordinate axes, i.e. $E = \{z = x + \imath y, xy = 0\}$. Describe all entire functions f satisfying $f(E) \subset E$.*

Proof: It is clear that functions of the form $f(z) = \alpha z^n e^{k\imath\pi/2}$ for $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$. If $z \in E$, in polar form we can write all such functions as

$$f(r, \theta) = r^n e^{n\imath\pi/2}.$$

If $z \in E$, this is rotation by a multiple of $\pi/2$ or a dialation, hence $f(z) \in E$ if $z \in E$. Now linear combinations will work if and only if the each exponent in the linear combintation is equal mod 2, i.e each term in the polynomial produces either a real or imaginary number. (e.g. the function $f(z) = z^4 + z^2 + 1 \in E$ if $z \in E$.

Exercise 2.16. *Let $p(x)$ be a polynomial with real coefficients. Prove that Laplace's transform*

$$\tilde{p}(z) = \int_0^\infty e^{-xz} p(x)\ dx,$$

is an analytic function in the right half-plane $\Re(z) > 0$.

Proof: First suppose that $p(x) = ax^n, for n \in \mathbb{N}$, then we have

$$\tilde{p}(z) = a\frac{n!}{z^{n+1}}$$

Now this function has a pole at 0, and hence is analytic for $\Re(z) > 0$. Integration is linear hence given a general polynomial

$$p(x) = \sum_{k=0}^{n} a_k x^k \quad \Rightarrow \quad \tilde{p}(z) = a_n \sum_{k=0}^{n} \frac{k!}{z^{k+1}}$$

which is a sum of analytic functions for $\Re(z) > 0$ and hence is analytic.