AN OVERVIEW OF STATIC HAMILTON-JACOBI EQUATIONS

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Abstract. There is a voluminous amount of literature on Hamilton-Jacobi equations. This paper reviews some of the existence and uniqueness techniques for the time independent cases with a particular interest in Eikonal-like Equations. In addition, key features of these equations, that have helped develop numerical schemes are noted.

1. Introduction

This paper's topic is the static Hamilton-Jacobi equation. It covers known methods for existence and uniqueness for solutions. First the equation of interest is derived from the optimality principle, then the method of characteristics, viscosity solutions and the adjoint method are discussed. Through out this paper we will use Eikonal equations as the primary example. Eikonal equations are found in a variety of applications from computational geometry, optics, image processing and fluid dynamics. A key feature of the Eikonal equation is that the trajectory of characteristic lines coincide with the gradient lines of the viscosity solution. This has many advantages for numerical approximations of solutions and has helped with the development of Dijkstra-like algorithms; such as, the level set method and ordered up winding methods. Algorithms of this form allow for approximating solutions to a wide class of static Hamilton-Jacobin equations. In general, a static Hamilton-Jacobi equation with boundary conditions is of the form

$$\begin{cases} H(x, u(x), \nabla u(x)) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \tag{1.1}$$

For this paper we will only consider the case where the Hamiltonian is independent of $u(x)$, i.e.,

$$\begin{cases} H(x, \nabla u(x)) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \tag{1.2}$$

Usual assumptions that go along with this PDE are: Ω is an open, bounded domain in $\mathbb{R}^n$. $g(x)$ is a positive, Lipschitz continuous function. Another useful assumption of this PDE is if the Hamiltonian H is homogeneous of degree one in the gradient argument. In this case, we have we have

$$H(x, \nabla u(x)) = \|p\| F\left(x, \frac{\nabla u(x)}{\|\nabla u(x)\|}\right) \tag{1.3}$$

for some function F. This property is convenient since it allows up to consider the gradient on unit ball in $\mathbb{R}^n$. Additionally, if the Hamiltonian H independent of it position we come to Eikonal-like equations, which are of the form:

$$\begin{cases} H(\nabla u(x)) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \tag{1.4}$$

Even if we place the conditions that $H(x, \nabla u(x)), g(x)$, and $\partial\Omega$ are smooth, a smooth solution $u(x)$ might not exist in Ω. In general, there are infinitely many weak Lipschitz continuous solutions; however, the unique viscosity solution can be defined usually additional conditions on test functions. The adjoint method compliments the method of vanishing viscosity, and points out an alternative view of approximating solutions, in turn which may help to develop efficient numerical schemes to this class of static Hamilton-Jacobi equations.

2. General Notation and Assumptions

For the sake completeness and brevity later, all general notation and assumptions used will be defined here. The abbreviation HJE will stand for the obvious Hamilton-Jacobi equation.

• As mentioned above, Ω will be an open, bounded domain in $\mathbb{R}^n$, with $\partial\Omega$ being smooth.

• $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$, i.e., $\mathbb{B} = \{z \in \mathbb{R}^n : |z| = 1\}$. This will be used as the set of admissible control values, while the set $\mathcal{A} = \{a : \mathbb{R}^+ \to \mathbb{B} : a(\cdot)$ is measurable $\}$ will denote the set of admissible controls. $\mathcal{D}(\Omega)$ denote the space of test functions on Ω

• p will represent the gradient of u ($p = \nabla u(x)$), and will be used interchangeably. Subscripted letters will represent fixed points or constants.

• u, g will be a functions independent of time, $u, g : \Omega \to \mathbb{R}$. As mentioned above, $g(x)$ will be a a positive, Lipschitz continuous function and it will be assumed that $g(x)$ is bounded and bounded away from zero on $\partial\Omega$, i.e., there are constants g_1 and g_2 such that

$$0 < g_1 \leq g(x) \leq g_2 < \infty$$

• K will be the map $K : (\Omega \times \mathbb{B}) \to \mathbb{R}^+$, and will represent some cost function. Where R^+ are the positive real numbers. Furthermore it will be assumed that K is bounded, and bounded away from zero, i.e., there are constants K_1 and K_2 such that

$$0 < K_1 \leq K(x, p) \leq K_2 < \infty$$

One final note, the assumptions on the Hamiltonian H will change throughout the paper, and will be listed as needed.

3. Derivation of the equation from the optimality principle

In physics, the HJE is a reformulation of classical mechanics. Mathematically this amounts to the necessary condition describing extremal geometry in calculus of variations problems. Some formulations can be found here [7],[9]. One of particular interest is the derivation from the Hamilton-Jacobi-Bellman (HBJ) equation. This a PDE which plays a dominant role in optimal control theory. Consider the problem of finding an optimal trajectory for an object moving with unit speed:

$$\begin{cases} y'(t) = a(t) \\ y(0) = x \qquad x \in \Omega \end{cases} \tag{3.1}$$

where $y(t)$ is the position of the object at time t. To extract a HJE from this problem, consider Bellman's optimality principle [2]. Let $\epsilon > 0$ be sufficiently small, then we have:

$$\begin{aligned} u(x) &= \inf_{a\in\mathcal{A}} \left\{ \int_0^\epsilon K(x(t),a(t)) + u(a(t))\ dt \right\} \\ &= \inf_{a\in\mathcal{A}} \left\{ \int_0^\epsilon K(x(0)+O(\epsilon),a(0)+O(\epsilon)) + u(x+\epsilon a(0)+O(\epsilon^2))\ dt \right\} \\ &= \min_{a\in\mathbb{B}} \left\{ \epsilon K(x,a) + u(x+\epsilon a) + O(\epsilon^2) \right\} \\ &= \min_{a\in\mathbb{B}} \left\{ \epsilon K(x,a) + (\nabla u(x)\cdot a) + O(\epsilon) \right\} + u(x) \end{aligned}$$

More details about Bellman's optimality principle can be found here. [1],[2]. Since the above hold for any small ϵ, we have that $u(x)$ should satisfy:

$$\begin{cases} \min\limits_{a\in\mathbb{B}} \{K(x,a) + \nabla u(x)\cdot a\} = 0 & x\in\Omega \\ u(x) = g(x) & x\in\partial\Omega \end{cases} \tag{3.2}$$

This equation is known as the HJB PDE. Now consider the following Hamiltonian:

$$H(x,p) = \min_{a\in\mathbb{B}} \{K(x,a) + p\cdot a\} \tag{3.3}$$

Hence we have just derived the familiar equation of $H(p,x) = 0$. Now there are two things to remark about this choice of Hamiltonian $H(x,p)$, it is not convex nor is it homogeneous of degree one in p. The condition of convexity on H will be needed for the viscosity solution. Convexity can be realized with a different but equivalent choice of Hamiltonian H, this will be done in section 6. The next problem becomes how do we solve, if possible, such an equation $H(x,\nabla u(x)) = 0$.

4. The Method of Characteristics

Given the nature of non-linearity, direct methods of solving the HJE is generally not possible. The method of characteristics is an indirect technique to find a local solution emanating from $\partial\Omega$. Consider the static HJE in (1.2). This equation can be solved locally by the method of characteristics. Unfortunately, for most cases a global solution is not possible. This is due to shock formations from characteristics crossing. Now consider a special case of (1.2):

$$\begin{cases} H(x,\nabla u(x)) = 0 & x\in\Omega \\ u(x_1,x') = g(x') & x = (x_1,x')\in\partial\Omega \end{cases} \tag{4.1}$$

For the method of characteristic of the static HJE, we're considering $x = (x_1,x')$ where x_1 is thought of as t for the time independent case. Then the part of hyperplane $x = (x_1,x') \subset \partial\Omega$, can be thought of as the initial condition. In order to solve (4.1) in terms of characteristics we need a couple conditions. First

$$\partial_{x_0} H(x,\nabla u(x)) \neq 0 \quad x = (x_1,x'). \tag{4.2}$$

We must also clearly assume $H(x,\nabla u(x)) = 0$, for $x = (x_1,x')$. Let γ be a curve emanating from $\partial\Omega$ with parameter s. Then suppose the arguments of the solution in (4.1) run along γ; that is, $x = x(s)$,

$p = p(s)$. This lead us to the following system of ODEs:

$$\begin{cases} \dot{x} = \nabla_p H(x(s), p(s)) \\ \dot{p} = -\nabla_x H(x(s), p(s)) \end{cases} \quad \text{with initial conditions} \quad \begin{cases} x(0) = x_1 \\ p(x_1) = \nabla u(x_1) \end{cases} \tag{4.3}$$

Now even before we have a solution $u(x)$, we know $\nabla u(x)$ for $x = (0, x')$ due to our equation for H. Using the our hypothesis 4.2, these equations have a local solution for some interval $0 \leq s < s_1$, this follows from standard ODE theorems. Hence, these curves fill out a neighborhood around $x = (x_1, x')$. This implies that the curves define the value of p in this neighborhood about the $\partial\Omega$. Now observe by the chain rule that:

$$\frac{d}{ds} H(x(s), p(s)) = 0 \quad \Rightarrow \quad H = 0 \text{ along } \gamma$$

Now we need our solution u to satisfy $\nabla u = p$, i.e., for every s, $\nabla u(x(s)) = p(x(s))$. Assuming such a solution exists, $u(x)$ must satisfy the following:

$$\frac{d}{ds} u(x(s)) = \nabla u(x(s)) \cdot \dot{x}(s) = p \cdot \frac{\partial H}{\partial p}(x(s), p(x(s))) \tag{4.4}$$

and so

$$u(x(s)) = u(x(0)) + \int_0^s p(t) \cdot \dot{x}(t)\ dt.$$

In other words, the solution u will be given in a neighborhood of $\partial\Omega$ by an explicit equation. Since the different paths $x(s)$, starting from different datum may cross, the solution may become multi-valued, these points are commonly referred to as shocks. Notice from the above we also have

$$p(x(s)) = p(x_1) - \int_0^s \nabla_x H(x(s), p(s)). \tag{4.5}$$

It remains to show that p, is the gradient of some function u in a neighborhood of $\partial\Omega$. This will follow from elementary calculus if we show that the vector field $p(x(s))$ is curl free. Consider the first term in the definition of p. This term, $p(x(0)) = \nabla u(x(0))$ is curl free as it is the gradient of a function. As for the mixed terms we have

$$\frac{\partial^2}{\partial x_k \partial x_j} H = \frac{\partial^2}{\partial x_j \partial x_k} H$$

and so we have a local solutions for $u(x)$ around $\partial\Omega$. [1]

If we look back at the control problem and formulation for the HJB PDE (3.2), there is a something nice to remark here.

Remark 1. *The direction of the characteristics $y \in \mathbb{B}$ for some point $x \in \Omega$ is the minimizer and $K(x, y) + \nabla u(x) \cdot y = 0$. Hence the characteristic directions of the PDE* (4.1) *are exactly the optimal control values and the characteristics lines are the optimal trajectories for the control problem.*

5. Viscosity Solution

Definition. *A bounded, uniformly continuous function $u(x)$ is the viscosity solution of the HJB PDE if the following holds for each $\phi \in \mathcal{D}(\Omega)$*

(i) if $u - \phi$ has a local maximum at $x_0 \in \Omega$ then

$$\min_{a \in B} \{\nabla\phi(x_0) \cdot a + K(x_0, a)\} \geq 0 \tag{5.1}$$

[1] As for an idea for the limit of the method of characteristics, consider the crossing of rays of light reflecting off of a concave mirror.

(ii) if $u - \phi$ has a local minimum at $x_0 \in \Omega$ then

$$\min_{a \in B} \{\nabla\phi(x_0) \cdot a + K(x_0, a)\} \leq 0 \tag{5.2}$$

Conditions (5.1) and (5.2) are referred to as super and sub solutions respectively. It is well know that there exists a unique viscosity solution to the HJB PDE [4],[5],[7]. That solution is Lipschitz continuous and, hence differentiable almost everywhere in Ω. We will show uniqueness for Eikonal-Like equations in section 7. If $\nabla(x_0)$ is defined then the function is also a solution in the classical sense, i.e.,

$$\min_{q \in B} \{K(x_0, q) + \nabla u(x_0) \cdot q\} = 0 \tag{5.3}$$

It can be rigorously shown that the function $u(x)$ of the optimal trajectory problem satisfies the conditions (5.1),(5.2) and is the unique viscosity solution of HBJ PDE [1]. Uniqueness will be shown for an Eikonal-like equations in section 7.

Conditions (5.1),(5.2) regulate how the solution behaves wherever ∇u is undefined, as mentioned above these are precisely the shocks of the characteristics. Whenever ∇u is well defined, the local extrema conditions would the test functions imply that $\nabla\phi = \nabla u$; thus, the equality would be attained for discontinuity by passing the value at the shocks to the test functions. These two inequalities specify a criterion similar to the entropy condition for the hyperbolic conservation laws: the characteristics can collide, but never emanate from the shock [7].

Remark 2. *the term viscosity solution refers to the fact that u can also be obtained by the method of vanishing viscosity, as a uniform limit:*

$$u(x) = \lim_{\epsilon \to 0} u_\epsilon(x)$$

where u_ϵ is the smooth and unique solution of the regularized equation

$$H(\nabla u_\epsilon(x), x) = \epsilon \Delta u_\epsilon(x).$$

For the moment, suppose that the Hamiltonian H is convex. Let's turn to the important and special case that the cost function K is isotropic, that is, $K(x, \nabla u(x)) = K(x)$ or depends only on it's location. In this case we have

$$\begin{aligned} 0 &= \min_{a \in \mathbb{B}} \{K(x) + \nabla u(x) \cdot a\} \\ &= K(x) + \min_{a \in \mathbb{B}} \{\nabla u(x) \cdot a\} \\ &= K(x) - \|\nabla u(x)\|^2 \end{aligned}$$

Thus we arrive at a familiar Eikonal equation:

$$\begin{cases} \|\nabla u(x)\|^2 = K(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \tag{5.4}$$

One nice property of this equation, which can be immediately seen from this derivation is the content of Remark 1. That is, $a = -\nabla u / \|\nabla u\|$ is the minimizer. Hence, the gradient lines coincide with the characteristics. This property is plays a crucial role in the development of numerical algorithms.

Remark 3. *The Eikonal equation (5.4) is of type $H(x,p) = 0$, so the work done for the method of characteristics directly applies.*

6. Equivalent Control Problem

In section 3 a class of HJEs was derived from the optimality principle. In using the optimality principle we have described a problem of finding an optimal trajectory for a function with a unit speed [2]. From this derivation the Hamiltonian was not convex nor homogeneous of degree one in the gradient argument. We will now show that this problem is equivalent to finding an optimal trajectory for an object with changing speed but constant running costs [1]. Looking back at the initial motivation (3.1), consider an object such that its position in Ω is given by

$$\begin{cases} v'(r(t)) = \dfrac{a(t)}{K(y(t),a(t))} = F(a(t))a(t) \\ v(0) = x \in \Omega \end{cases} \tag{6.1}$$

where $r(t) = \displaystyle\int_0^t K(y(s),a(s))ds$. Then we have

$$\begin{aligned} \frac{d}{dt}\left(v(r(t)) - y(t)\right) &= v'(r(t) \cdot r'(t) - y'(t) \\ &= \frac{a(t)}{K(y(t),a(t))} \cdot K(y(t),a(t)) - a(t) = 0 \end{aligned} \tag{6.2}$$

Hence we have $y(t) = v(r(t))$. Now $r'(t) > 0$, and so $r(t)$ is invertible. Implementing this with $\bar{a}(r) = \bar{a}(t(r))$ into (6.1) we have

$$v'(r) = f(\bar{a}(r))\bar{a}(r) = \frac{\bar{a}(r)}{K(v(r),\bar{a}(r)} \tag{6.3}$$

Now considering (6.3) with running cost equal 1 we come to an equivalent equation of

$$u(x) = \inf_{a\in\mathcal{A}}\{C(x,a)\} = \inf_{\bar{a}\in\mathcal{A}}\{\bar{C}(x,\bar{a})\} \tag{6.4}$$

where C, $\bar{C}$ represent the cost using controls a, $\bar{a}$, respectively [2]. This implies that $u(x)$ is also the value function corresponding the the optimal control function (3.2). As such it has to be the unique viscosity solution of the corresponding HJB equation.

$$\min_{a\in\mathbb{B}}\left\{\nabla u(x) \cdot \frac{a}{K(x,a)}\right\} = \min_{a\in\mathbb{B}}\left\{(\nabla u(x) \cdot a)f(x,a)\right\} = 0 \tag{6.5}$$

Now if we choose the Hamiltonian

$$H(x,p) = -\min_{a\in\mathbb{B}}\left\{(p \cdot a)f(x,a)\right\} = \max_{a\in\mathbb{B}}\left\{p \cdot (-a))f(x,a)\right\} \tag{6.6}$$

then we arrive at the equation (1.2). Furthermore, this Hamiltonian is convex and homogeneous of degree 1 in the gradient argument. Now equations (3.2) and (6.6) solve the same control problem under different assumptions. For each case, the cost function K is bounded, so using our bounds we have

$$0 < K_2^{-1} = f_1 < f < f_2 = K_1^{-1}$$

and thus we come to the anisotropy coeffic For another important observation for numerical schemes, define

$$\mathbb{B}(\phi) = \{a \in \mathbb{B} : a \cdot \nabla\phi \leq -\|\nabla\phi(x)\|\frac{K_1}{K_2} \tag{6.7}$$

refer back to the definition of viscosity solution of our new Hamiltonian we have:

Definition. *A bounded, uniformly continuous function $u(x)$ is the viscosity solution of the (6.5) if the following holds for each $\phi \in \mathcal{D}(\Omega)$*

(i) if $u - \phi$ has a local maximum at $x_0 \in \Omega$ then

$$\min_{a \in B} \left\{ \nabla\phi(x_0) \cdot \frac{q}{K(x_0, a)} \right\} \geq 0 \tag{6.8}$$

(ii) if $u - \phi$ has a local minimum at $x_0 \in \Omega$ then

$$\min_{a \in B} \left\{ \nabla\phi(x_0) \cdot \frac{a}{K(x_0, a)} \right\} \leq 0 \tag{6.9}$$

By replacing $\mathbb{B}$ with $\mathbb{B}(\phi)$ we have an equivalent definition. Now if the minimum is attained for some $a = a_1$, then we have $a_1 \cdot \phi(x_0) < 0$, and so

$$\nabla\phi(x_0) \cdot \frac{a_1}{K(x_0, a_1)} \geq \nabla\phi(x_0) \cdot \frac{a_1}{K_1} \tag{6.10}$$

Let $b = -\dfrac{\nabla\phi(x_0)}{\|\nabla\phi(x_0)\|}$. Since a_1 is the minimizer we have

$$\nabla\phi(x_0) \cdot \frac{a_1}{K(x_0, a_1)} \leq \nabla\phi(x_0) \cdot \frac{b}{K(x_0, b)} \leq -\frac{\|\nabla\phi(x_0)\|}{K_2}$$

and so we have

$$a_1 \cdot \nabla\phi(x_0) \leq -\|\nabla\phi(x_0)\| \frac{K_1}{K_2} \tag{6.11}$$

Thus we have just established a bound on the angle between the characteristics and the gradient line. If the gradient $\nabla u(x_0)$ exists then $\nabla u(x_0) = \nabla\phi(x_0)$, and thus

$$a_1 \cdot \nabla u(x_0) \leq -\|\nabla u(x_0)\| \frac{K_1}{K_2} \tag{6.12}$$

Therefore if α is the angle between $\nabla u(x_0)$ and $-a_1$ then $\cos(\alpha) \geq \dfrac{K_1}{K_2}$. This observation is important in the construction of single-pass methods for HJE of this type.

7. Existence and Uniqueness of the Viscosity Solution for Eikonal-like equations

Now that we have done a few computations regarding Eikonal-like equations from the optimality principle, let's turn to the question of existence and uniqueness. For existence and uniqueness, we need the following assumptions on the Hamiltonian H.

(H1) H is smooth and $H(0) < 0$

(H2) H is coercive, that is $\lim_{|p| \to \infty} \dfrac{H(p)}{p} = \infty$

(H3) H is convex

First, for existence of the viscosity solution for the Eikonal-like equation (1.4), let us consider the regularized equation

$$\begin{cases} H(\nabla^\epsilon u(x)) = \epsilon \Delta u^\epsilon(x) & x \in \Omega \\ u^\epsilon(x) = g(x) & x \in \partial\Omega \end{cases} \tag{7.1}$$

It can be shown from elliptic regularity [8] that there exists $C > 0$ such that

$$\|u^\epsilon\|_\infty, \|\nabla u^\epsilon\|_\infty \leq C$$

The existence of the solution follows immediately from these estimates, as the uniform limit of viscosity solution is a viscosity solution. For uniqueness of the viscosity solution for equation (1.4), we start with a more general lemma

Lemma: (L7.1) *Let $f \in C(\Omega)$ satisfy $f(x) < 0$ for $x \in \Omega$, and $u, v \in C(\Omega)$ satisfy*

$$H(x, \nabla u) \leq f(x) \text{ and } H(x, \nabla v) \geq 0 \qquad \text{for } x \in \Omega$$

in the viscosity sense. If $u(x) \leq v(x)$ on $\partial\Omega$, then $u(x) \leq v(x)$ in Ω.

Proof: By theorem 2.1 in [4] we have $\sup_{x\in\Omega}(u(x) - v(x)) \leq \sup_{x\in\Omega} f(x)$. If $\sup_{x\in\Omega}(u(x) - v(x)) > 0$, then we have a contradiction to $f(x) < 0$, so the result is shown. $\square$

Theorem: (T7.1) *Let $u(x), v(x) \in C(\bar{\Omega})$ satisfy*

$$H(x, \nabla u(x)) \leq 0 \quad \text{and} \quad H(\nabla u(x)) \geq 0 \quad x \in \Omega$$

in the viscosity sense. Also suppose that $u(x) \leq v(x)$ on $\partial\Omega$. Then $u(x) \leq v(x)$ on Ω

Proof: Let $\theta \in (0, 1)$ and define u_θ as follows:

$$u_\theta(x) = \theta u(x) + (1 - \theta)\phi(x) \qquad \text{for } x \in \bar{\Omega}$$

we can choose $f(x) \in C(\bar{\Omega})$ such that $H(\nabla\phi(x)) \leq f(x) < 0$ for $x \in \Omega$. (WLOG) we can assume that $\phi \leq u(x)$ on Ω, otherwise use a simple translation $\phi_M(x) := \phi(x) - M$, for some $M > 0$. Then we have $u_\theta(x) \leq u(x)$ on Ω and $u_\theta(x) \in C(\bar{\Omega})$. By convexity and a simple computation we have

$$\begin{aligned} H(x, \nabla u_\theta) &\leq \theta H(x, \nabla u) + (1 - \theta)H(x, \nabla\phi) \\ &\leq \theta H(x, \nabla u) + (1 - \theta)f(x) \leq (1 - \theta)f(x) \end{aligned}$$

So we have $H(x, \nabla u_\theta) \leq (1 - \theta)f(x)$ in the viscosity sense. Thus $u_\theta(x) \leq v(x)$ for all $0 < \theta < 1$ by Lemma (L7.1). Therefore $u(x) \leq v(x) \in \Omega$ $\square$

Now for uniqueness of the Eikonal-Like equation (1.4) consider the following function:

$$\phi(t) = t^{-\gamma} H(tp), \forall t > 0, \text{ where } \gamma > 0 \tag{7.2}$$

then we have

$$\begin{aligned} \frac{d}{dt}\phi(t) &= -\gamma t^{-\gamma-1} H(tp) + t^{-\gamma} H(tp) \cdot p \\ &= t^{-\gamma-1} \left(\nabla H(tp) \cdot (tp) - \gamma H(tp)\right) \end{aligned} \tag{7.3}$$

We want $\phi(t)$ to be strictly increasing for $t < 1$. To impose this condition this, suppose there is a $\delta > 0$ such that

$$\nabla H(tp) \cdot (tp) - \gamma H(tp) > \delta > 0 \tag{7.4}$$

Then looking at (7.3) we have

$$\phi'(t) = t^{-\gamma-1}\left(DH(tp) \cdot (tp) - \gamma H(tp)\right) > t^{-\gamma-1}\delta > 0$$

so we can write

$$\phi(1) - \phi(t) = \int_t^1 \phi'(s)ds > \int_t^1 s^{-\gamma-1}\delta ds = \frac{\delta}{\gamma+1}(t^{-\gamma} - 1) > 0$$

Hence we have

$$H(tp) \leq t^\gamma H(p) - \frac{\delta}{\gamma+1}(1 - t^\gamma) = t^\gamma H(p) + \frac{-\delta}{(\gamma+1)H(0)}(1 - t^\gamma)H(0)$$

So H satisfies all the conditions with $\phi = 0$, hence the uniqueness of the viscosity solution follows by (T7.1). [2]

Remark 4. *Notice that we can relax the convexity condition slightly and replace it with the following condition*

(H3') $\nabla H(tp) \cdot (tp) - \gamma H(tp) > \delta > 0$

This condition will appear in while considering the adjoint equation.

8. Adjoint method for Eikonal-like equations

In general, adjoint equations are Linear PDEs derived from a PDE of interest. Gradient value with respect to a a particular quantity of interest can be efficiently calculated by solve the adjoint equation. No methods for solving the adjoint equation will be done here, instead we list properties and relations to the original equation. Considering the regularized equation (7.1), let $w^\epsilon = \dfrac{\|\nabla u^\epsilon\|^2}{2}$.

Lemma (L8.1) w^ϵ *satisfies:*

$$\nabla H(\nabla u^\epsilon) \cdot \nabla w^\epsilon = \epsilon \Delta w^\epsilon - \epsilon |D^2 u^\epsilon|^2. \tag{8.1}$$

Proof: Let $\epsilon > 0$, and consider the regularized equation (8.1). For the sake of notation let $p^\epsilon = \nabla u^\epsilon$, differenang (8.1) with respect to x_i we have

$$H_{p_k^\epsilon}(u^\epsilon_{x_k x_i}) = \epsilon \Delta u^\epsilon_{x_i} \tag{8.2}$$

Multiplying (8.2) by u_{x_i} and summing over i come to the expression

$$D_{p^\epsilon} H(p^\epsilon) \cdot D_x\left(\frac{|p^\epsilon|^2}{2}\right) = \epsilon \sum_{i=1}^n (\Delta u^\epsilon_{x_i}) u^\epsilon_{x_i} \tag{8.3}$$

[2] The proof for the uniqueness of u^ϵ is omitted but follows a similar ideas of section 7 and can be found here [14].

Working with the terms $\Delta u^\epsilon_{x_i} u^\epsilon_{x_i}$ inside the sum of the right hand side we have

$$\begin{aligned}\Delta u^\epsilon_{x_i} u^\epsilon_{x_i} &= u^\epsilon_{x_k x_k x_i} u^\epsilon_{x_i} \\ &= (u^\epsilon_{x_k x_i} u^\epsilon_{x_i}) x_k - \sum_{k,l} |u^\epsilon_{x_k x_l}|^2\end{aligned}$$

Now summing over i implies

$$\epsilon \sum_{i=1}^{n} (\Delta u^\epsilon_{x_i}) u^\epsilon_{x_i} = \Delta \left(\frac{|\nabla u^\epsilon|^2}{2} \right) - |D^2 u^\epsilon|^2 \tag{8.4}$$

Plugging w^ϵ into (8.3) and (8.4) we have the claim of the lemma.

For each fixed $x_0 \in \Omega$, consider the following PDE:

$$\begin{cases} -\nabla \cdot (\nabla H(\nabla u^\epsilon) z^\epsilon) = \epsilon \Delta z^\epsilon + \delta_{x_0} & x \in \Omega \\ \sigma^\epsilon = 0 & x \in \partial\Omega \end{cases} \tag{8.5}$$

From the solution of the adjoint equation, z^ϵ, we can observe properties of u^ϵ as well as u. The first thing to observe about if $z^\epsilon = w^\epsilon$ in (8.5) is that by expanding we have

$$-\Delta H(\nabla u^\epsilon)(\Delta u^\epsilon) w^\epsilon - \nabla H(\nabla u^\epsilon) \cdot \nabla w^\epsilon = \epsilon \Delta w^\epsilon + \delta_{x_0} \tag{8.6}$$

adding (8.1) and (8.6) together we have

$$-\Delta H(\nabla u^\epsilon)(\Delta u^\epsilon) w^\epsilon = \epsilon |D^2 u^\epsilon|^2 + \delta_{x_0} \tag{8.7}$$

Analysis of (8.5) is interesting in it's own right, we're going to list a few properties about σ^ϵ with out proof. The proof of these properties can be found [14].

(A1) $z^\epsilon \geq 0$ in $\Omega/\{x_0\}$. Which implies $\partial_\nu \sigma^\epsilon \leq 0$ on $\partial\Omega$

(A2) $\epsilon \int_{\partial\Omega} \partial_\nu z^\epsilon d\sigma_\Omega = -1$

(A3) There exists a constant C such that $\epsilon \int_\Omega |D^2 u^\epsilon|^2 z^\epsilon \; dx \leq C$.

The next section will illustrate the benefits and problems that can arise when considering the adjoint equation

9. Examples Regarding Adjoint Equations

First consider a simple but non-trivial example when $\nabla H(p) = 0$. Such an example of an adjoint equation on $\mathbb{R}$ looks like

$$\begin{cases} \epsilon v^\epsilon_{xx} + 1 = 0 & x \in (0,1) \\ v^\epsilon(0) = v^\epsilon(1) = 0 \end{cases} \tag{9.1}$$

solving this ODE implies

$$v^\epsilon(x) = \frac{1}{2\epsilon}(x - x^2) \quad \Rightarrow \quad \max_{x \in [0,1]} v^\epsilon = \frac{1}{8\epsilon} \tag{9.2}$$

so $\max v^\epsilon$ blows up as ϵ tend to 0. This example shows that we need to conditions on the gradient of the Hamiltonian H that allows for control of v^ϵ. Consider the slightly more complicated example where

$\nabla H(p) = -p$. With this choice of Hamiltonian we come to the equations

$$\begin{cases} v^\epsilon_x = \epsilon v^\epsilon_{xx} + 1 & x \in (0,1) \\ v^\epsilon(0) = v^\epsilon(1) = 0 \end{cases} \tag{9.3}$$

Solving this ODE we have

$$v^\epsilon(x) = x - \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1} \quad \Rightarrow \quad \max_{x \in [0,1]} v^\epsilon \le 1 \ \forall \epsilon > 0 \tag{9.4}$$

Hence we have a uniform bound on v^ϵ independent of ϵ. Considering the two examples above we consider again the condition (H3'). If we choose $\gamma = 1$ and $\delta = -H(0)$, then we can the relax convexity condition

$$\nabla H(tp) \cdot (tp) - \gamma H(tp) \ge -H(0) > 0 \quad \forall p \in \mathbb{R}^n \tag{9.5}$$

Now notice that if the Hamiltonian H is convex then the condition (H3') follows from the above choice of γ and δ. For a simple example of this relaxed convexity condition, consider $H(p) = (p^2-1)^2 - 2 = p^4 - 2p - 1$. $H(p)$ is not convex however $\nabla H(p) \cdot p - 2H(p) = 2p^4 + 2 \ge 2 > 0$

In summary, it is enough to require conditions (H1), (H2), (H3') on the Hamiltonian to guarantee a unique solution for the adjoint equation [6], [14].

10. Final Remarks

The class of HJEs this paper considered have convenient properties for the development of numerical schemes.

- Considering the problem of finding an optimal trajectory for an object with unit speed we have the gradient lines coincide with the characteristics.

- For the similar control problem of changing speed but constant cost, we have a bound on the angle between the gradient and the characteristics.

On a final note, there has been little progress in the last 20 years for understanding gradient shock structures of non-convex Hamiltonians. The adjoint method introduced by Evans recently [6] gives us a new technique for analyzing a class of these non-convex HJEs. In turn, interesting results may appear in adapting numerical schemes for the adjoint equations corresponding to the regularized equations of the Hamilton-Jacobi equation.

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