Jarrod Pickens

Research Statement

My research interests lie primarily in the fields of differential geometry and geometric analysis. Namely, I am interested in studying geometric flows and investigating their applications to geometry, topology, and mathematical physics. Geometric flows have been studied extensively in recent years, most notably the Ricci flow, which has been used to prove Thurston's geometrization conjecture and the Poincaré conjecture. The Ricci flow itself is not completely understood, and there remains many interesting questions to answer. I am currently studying the Ricci flow of manifolds which admit circle actions. Below I will outline my current research project and my plans for the immediate future.

Given a Riemannian manifold (M, g_0), the Ricci flow produces a one-parameter family of metrics $g(t)$ on M, determined by the following system of partial differential equations:

$$\partial_t g(t) = -2\mathrm{Rc}(g(t))$$
$$g(0) = g_0$$

where $\mathrm{Rc}(g)$ is the Ricci curvature of $g(t)$. This was introduced by Richard Hamilton to study 3-manifolds of positive Ricci curvature. He proved that if the initial metric has positive Ricci curvature, then under an appropriate normalization, the flow will converge in finite time to a metric of constant positive curvature [1]. For an arbitrary initial metric, the flow will always last for a short time, but finite time singularities may form which cause the flow to terminate. Ricci flow with surgery has been introduced to deal with the formation of singularities. Typically, the flow is stopped shortly before a singularity occurs, and a topological surgery is performed on the manifold such that when the flow is restarted, the singularity does not form. Recently, the Ricci flow (with surgery) has been used by Grisha Perelman to prove Thurston's geometrization conjecture and hence the Poincaré conjecture [10],[8],[9] (For more information, see also [4],[2]).

With the success of the program above, it is desirable to discover all that we can about the Ricci flow. In my current research, I study the Ricci flow on manifolds which admit an S^1-action. In particular, I am examining the formation of singularities and large-time behavior (using blowdown limits) of the Ricci flow for such manifolds. These manifolds are interesting in the context of the geometrization conjecture and also string theory. Let M be a manifold which admits a circle action, and let $\Sigma = M/S^1$ denote the orbit space. If M is a closed, orientable, $3-$dimensional manifold that admits an effective S^1-action, then Σ will be a compact $2-$dimensional manifold whose boundary consists of the fixed points of the action. Let g be a Riemannian metric on Σ. It is natural then to consider metrics of the form $f^2(x,y)dz^2 + g$ where (x, y) are coordinates on Σ and $f : \Sigma \to \mathbb{R}_{\geq 0}$ represents the "radius" of the orbit. As the boundary consists of fixed points, it is required that $f = 0$ only on the boundary of Σ. In higher dimensions, Σ becomes an orbifold. Near principal orbits and fixed points on the boundary of Σ, it is appropriate to consider metrics of the form $f^2(x_1, \dots, x_{n-1})dx_n^2 + g$, where $(x_1, \dots, x_{n-1})$ are coordinates on Σ where f is as above. A metric of this type (a warped product) produces the following Ricci flow equations:

$$\partial_t g = -2\mathrm{Rc}(g) + \frac{2}{f}\mathrm{Hess}_g(f)$$
$$\partial_t f = \triangle f$$

where $\mathrm{Hess}_g(f)$ is the Hessian of f and $\triangle f$ is the Laplacian of f, both with respect to the metric g.

In the case when Σ is closed manifold (and so the action has no fixed points), it can be shown that the Ricci flow exists until the Riemannian curvature of g blows up (so singularities occur on the base, not the fiber). Many other aspects of the Ricci flow "descend" to g in this case. For example, g is κ-non-collapsed (on appropriate space-time intervals and scales), and we can rescale the metric around a singularity to obtain an ancient solution of the Ricci flow (one which exists on a time interval $[-\infty, 0]$). With this information, one can perform the surgeries prescribed by Perelman on Σ.

I am currently working to extend my results above to the case when Σ has a boundary (and so the action has fixed points). This involves studying the Ricci flow on manifolds with boundary, which has been partially examined in [11]. The author has shown short-time existence for the Ricci flow on manifolds with boundary in the case when the boundary is umbilical and assuming Neumann conditions on the boundary. I have extended these results to a similar problem involving mixed boundary conditions. I am also working on extending the work to Dirichlet boundary conditions in order to obtain a more complete picture of the Ricci flow on manifolds with boundary. The conditions on f required to make the warped product metric above smooth at the boundary imply that the boundary of Σ is totally geodesic, and so short-time existence of the Ricci flow above is guaranteed. I am also working to study singularity formation in this case, where more complex behavior may be present than in the fixed point free case. For example, it is no longer clear where singularities form. Also, it may be the case that different components of fixed points may coalesce via a singularity. Once the Ricci flow is understood in these cases, it may be possible to acquire information about the flow on manifolds with more general circle actions, namely those with exceptional orbits described in [7].

In the future, I hope to apply this work to other problems. Since the Ricci flow is like a heat equation for the Riemannian metric, the metric should smooth out and in some sense the evolving manifold should approach the locally homogenous pieces described in the geometric decomposition. Perelman has shown that hyperbolic pieces appear asymptotically in the Ricci flow, and that the non-hyperbolic pieces become graph manifolds (which have a geometric decomposition)[5]. John Lott, in a recent paper [3], has shown that in certain situations the Ricci flow performs this geometric decomposition directly. I hope to be able to apply my research to extend these results. In another application, physicists have shown (using advances due to Perelman) that the renormalization group flow for certain quantum field theories are gradient flows [6], and that the Ricci flow is one of these RG flows. I would like to study how my work can be applied in this context. I am also interested in applying my knowledge of the Ricci flow on manifolds with boundary to other geometric flows. In particular I would like to study the Kähler-Ricci flow and cross-curvature flow on manifolds with boundary and their applications.

Bibliography

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