Practice Midterm Math 104B

Instructions: Write clearly and show all your work.

1. Find α so that

$$\mathbf{A} = \begin{bmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

is positive definite.

2. Find α and $\beta > 0$ so that

$$\mathbf{A} = \begin{bmatrix} 4 & \alpha & 1 \\ 2\beta & 5 & 4 \\ \beta & 2 & \alpha \end{bmatrix}$$

is strictly diagonally dominant.

3. Consider the tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a)Find Crout's factorization:

$$\mathbf{A} = \begin{bmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ 0 & l_4 & l_5 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Use the factorization found in (a) to solve $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = (3, -2, 0)^T$.

4. Consider the linear system:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -1, \\ 2x_1 + x_2 - 3x_3 &= 3, \\ x_1 - x_2 + x_3 &= 0, \end{aligned}$$

and let $\mathbf{x}^0 = \mathbf{0}$.
a) Do the first two iterations of Jacobi.
b) Do the first two iterations of Gauss-Seidel.
c) Do the first two iterations of S.O.R with $\omega = 1.2$.
d) Which of the three approximations is closer to the exact solution $(1, 1, 0)$?

5. Show that if A is SDD then $\|T_j\| < 1$.

6. Consider an iterative method of the form:

$$\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, \ldots$$

with $\|T\| < 1$ and $\mathbf{x}^{(0)}$ and $\mathbf{c}$ arbitrary. Prove that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$$

and

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

7. Consider the system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5 \end{aligned}$$

By finding the spectral radius of T_j and of T_g prove that the Jacobi method diverges while Gauss-Seidel's method converges for this system.

8. Consider the matrix Given the matrix

$$\mathbf{T} = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

a) Show that $\mathbf{T}$ is positive definite.
b) Compute the $\rho(\mathbf{T})$, the spectral radius of $\mathbf{T}$.
c) Suppose you have an iterative method defined by this particular matrix $\mathbf{T}$ in the form

$$\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 0, 1, \ldots$$

Will the iterations converge? explain.

9. When are iterative methods preferable to direct methods (i.e. Gaussian elimination)?

10. True (T) or False (F). Suppose A is an $n \times n$ symmetric, positive definite matrix:

    a) ( ) The vector $\mathbf{x}$ that minimizes $\mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$

    b) ( ) S.O.R. will converge for $0 < \omega < 2$.

    c) ( ) Gaussian elimination can be performed without row interchange.

    d) ( ) $\|A\|_2 = \det(A)$.

    e) ( ) Crout's method could be applied to solve $A\mathbf{x} = \mathbf{b}$.

11. Consider the linear system

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63} \tag{1}$$
$$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168}. \tag{2}$$

(a) Find the condition number $K_\infty(A)$ of the matrix of coefficients A in the infinity norm.
(b) Let $\tilde{\mathbf{x}} = (0.142, -0.166)^T$ be an approximation to the solution $\mathbf{x}$ of the system. Using $K_\infty(A)$ find an estimate of the relative error $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty / \|\mathbf{x}\|_\infty$.

12. True (T) or False (F). Suppose A is any $n \times n$ SDD matrix and $\mathbf{x}^{(0)}$ is arbitrary then:

a) ( ) Jacobi converges but Gauss-Seidel does not.

b) ( ) $\rho(T_\omega) \geq |\omega - 1|$.

c) ( ) S.O.R. will converge only for $\omega \geq 2$

13. Let $\tilde{\mathbf{x}}$ be an approximation to the solution $\mathbf{x}$ of the linear system $A\mathbf{x} = \mathbf{b}$. Prove that the error $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$ satisfies

$$A\mathbf{e} = \mathbf{r}$$

where $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$.

14. Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(a) Show that A is positive definite and find its condition number in the 2 norm, i.e., $K_2(A)$.
(b) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (0, 1)^T$. Taking $\mathbf{x}^(0) = (0, 0)^T$ as your initial guess, compute the first three iterations of the steepest descent method.

15. Write a 1/2-page summary of the main topics you have studied so far.