Optional Homework, Math 104B

Turn in a neatly written and well organized work. Include properly commented codes and printouts.

1. Let A be an $n \times n$ nonsingular matrix and $\mathbf{f}$ a given n-vector. Consider the linear system $A\mathbf{u} = \mathbf{f}$. (a) Show that an approximation $\mathbf{v}$ to the solution $\mathbf{u}$ has an error $\mathbf{e} = \mathbf{u} - \mathbf{v}$ that satisfy the residual equation $A\mathbf{e} = \mathbf{r}$, where $\mathbf{r} = \mathbf{f} - A\mathbf{v}$. (b) Show that $\mathbf{u} = \mathbf{v} + \mathbf{e}$.

2. Consider the boundary value problem:

$$-u'' + \pi^2 u = 5\pi^2 \sin(2\pi x) \quad 0 < x < 1, \tag{1}$$

$$u(0) = u(1) = 0. \tag{2}$$

   As discussed in class, we can find a numerical approximation to the solution of this problem by employing the finite difference method. Use a uniform grid with $N-1$ interior nodes to obtain, by replacing the second derivative with a second order finite difference and neglecting the (truncation) error, the linear system

$$\frac{-v_{j-1} + 2v_j - v_{j+1}}{h^2} + \pi^2 v_j = 5\pi^2 \sin(2\pi x_j) \quad \text{for } j = 1, 2, \ldots, N-1. \tag{3}$$

   where h=1/N, v_j is the approximation to $u(x_j)$ for $j = 1, 2, \ldots, N-1$, and $v_0 = v_N = 0$.
   (a) Use the Gauss-Seidel Method with a stopping criterium of 10^{-4} to find an approximation to the solution for $N = 64$ and for $N = 128$. Compute the actual error $\|u - v\|_\infty$, where u is the exact solution of the BVP (1)-(2), evaluated at the interior nodes.
   (b) Use the Conjugate Gradient Method with a stopping criterium of 10^{-4} to find an approximation to the solution for $N = 64$ and for $N = 128$. Compute the actual error $\|u - v\|_\infty$, where u is the exact solution of the BVP (1)-(2), evaluated at the interior nodes and compare to the result in part (a).

3. (a) Implement a coarse grid error correction strategy as in the multigrid, for solving the linear system (3). The scheme should have the following steps: 1) Apply Gauss-Seidel (GS) a few times (3-10) to the system that you get for a mesh size $h = 1/N$, $A^h\mathbf{u}^h = \mathbf{f}^h$ with a given initial guess of your choice. 2) Take the approximation $\mathbf{v}^h$ obtained with GS and compute the corresponding residual $\mathbf{r}^h = \mathbf{f}^h - A\mathbf{v}^h$. 3) Transfer $\mathbf{r}^h$ to the coarser grid with mesh size $2h = 1/(N/2)$ to obtain a $\mathbf{r}^{2h}$. The simplest way to do this is to transfer the points $r_j^{2h} = r_{2j}^h$ for $1 \le j \le N/2 - 1$. 4) Solve $A^{2h}\mathbf{e}^{2h} = \mathbf{r}^{2h}$ (e.g. with your Crout's solver). 4) Interpolate linearly $\mathbf{e}^{2h}$ onto the fine grid to obtain

an approximation to the error $\tilde{\mathbf{e}}^h$. 5) Correct the approximation: $\mathbf{v}^h \leftarrow \mathbf{v}^h + \tilde{\mathbf{e}}^h$.
(b) Apply the coarse grid correction to find an approximation to the solution of (3) for $N = 128$. Comment on the accuracy of the approximation and the cost to obtain it.
(c) Discuss how you could further reduce the cost of finding an accurate approximation by successive applying coarse grid error corrections. What about improving your initial guess(es)?