1. (*5 points*, no partial credit) True (T) or False (F). Consider the linear system $A\mathbf{x} = \mathbf{b}$. Then for **all** $n \times n$ (symmetric) positive definite matrices A with maximum eigenvalue $\lambda_{\max}$ and minimum eigenvalue $\lambda_{\min}$:

a) (T) The vector $\mathbf{x}$ that minimizes $\mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$

b) (F) The condition number in the 2 norm is $K_2(A) = \lambda_{\max}$.

c) (T) Let $\mathbf{r}$ be the residual of an approximation $\tilde{\mathbf{x}}$, then $\|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \|A^{-1}\|\|\mathbf{r}\|$.

d) (F) The condition number of A in the infinity norm, $K_\infty(A)$ satisfies $0 \leq K_\infty(A) < 1$.

e) (T) The search directions in the Steepest Descent Method are the residuals.

2. (*7 points*, hint: verify your answers) Consider the 3 × 3 tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

(a)Find Crout's factorization:

$$\mathbf{A} = \begin{bmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ 0 & l_4 & l_5 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Use the factorization found in (a) to solve $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = (2, 3, 3)^T$.

(a) $$\begin{bmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ 0 & l_4 & l_5 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_1u_1 & 0 \\ l_2 & l_2u_1+l_3 & l_3u_2 \\ 0 & l_4 & l_4u_2+l_5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$\boxed{l_1=2}$, $\boxed{l_2=1}$, $\boxed{l_4=1}$, $l_1u_1=2$, $\boxed{u_1=1}$ $l_2u_1+l_3=3$

$l_3 = 3-l_2u_1 = 2$, $\boxed{l_3=2}$ $l_3u_2=2$, $\boxed{u_2=1}$, $l_4u_2+l_5=3$

$l_5 = 3-l_4u_2 = 2$, $\boxed{l_5=2}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$\Rightarrow y_1 = 1$,

$y_1+2y_2=3$

$\Rightarrow y_2=1$

$y_2+2y_3=3$

$2y_3=2$

$y_3=1$

$\underset{\rightharpoonup}{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{x_3=1}$

$x_2+x_3=1$

$\boxed{x_2=0}$

$x_1+x_2=1$

$\boxed{x_1=1}$

$$\boxed{\underset{\rightharpoonup}{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

3. (*6 points*) Consider the linear system

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 = -\frac{1}{2} \qquad (1)$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 = -\frac{5}{12} \qquad (2)$$

(a) Show that the matrix of coefficients for this system

$$A = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}$$

is positive definite.
(b) Find the condition number $K_\infty(A)$ in the ∞-norm.
(c) Let $\tilde{\mathbf{x}} = (0.8, -2.9)^T$ be an approximation to the solution $\mathbf{x}$ of the system. Using $K_\infty(A)$ find an estimate of the relative error $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty / \|\mathbf{x}\|_\infty$.

(a) A is symmetric and

$\det(A_1) = \frac{1}{2} > 0$ $\det(A_2) = \det(A) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \frac{1}{3}\frac{1}{3} = \frac{1}{8} - \frac{1}{9} > 0$

i.e Principal submatrices have pos determinants

(b) $A^{-1} = 72 \begin{bmatrix} 1/4 & -1/3 \\ -1/3 & 1/2 \end{bmatrix}$, $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = \frac{5}{6}$

$\|A^{-1}\|_\infty = 72 \frac{5}{6}$ $\quad K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 12 \frac{25}{6} = 50$

$\boxed{K_\infty(A) = 50}$

(c) $\dfrac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \le K_\infty(A) \dfrac{\|r\|_\infty}{\|b\|_\infty}$, $\|b\|_\infty = \frac{1}{2}$ $\quad \|r\|_\infty = \|b - A\tilde{x}\|_\infty = 1/15$

$\dfrac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \le 50 \dfrac{1/15}{1/2} = \boxed{\frac{20}{3} \sim 6.6\overline{6}}$

4. (*6 points*) Consider the linear system

$$x_1 + x_2 = b_1 \tag{3}$$
$$x_1 + 2x_2 = b_2, \tag{4}$$

for given b_1 and b_2.
(a) Prove that Jacobi iteration converges for this system for any initial guess $\mathbf{x}^{(0)}$.
(b) Let $b_1 = 0$, $b_2 = -1$, and $\mathbf{x}^{(0)} = (1/2, -1/2)^T$. Perform two iterations of Jacobi.
(c) Let $b_1 = 0$, $b_2 = -1$, and $\mathbf{x}^{(0)} = (1/2, -1/2)^T$. Perform two iterations of Gauss-Seidel.

(a) Jacobi $x_1^{(k+1)} = b_1 - x_2^{(k)}$

$x_2^{(k+1)} = \dfrac{b_2 - x_1^{(k)}}{2}$

$$T_J = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$|T_J - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ -\frac{1}{2} & -\lambda \end{vmatrix} = \lambda^2 - \tfrac{1}{2} = 0 \quad \lambda = \pm \frac{1}{\sqrt{2}} \quad \rho(T_J) = \frac{1}{\sqrt{2}} < 1$$

Since $\rho(T_J) < 1 \Rightarrow$ Jacobi Converges

(b) $x_1^{(1)} = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$ | $x_1^{(2)} = 0 + \frac{3}{4} = \boxed{3/4}$

$x_2^{(2)} = \frac{1}{2}\left(-1 - \frac{1}{2}\right) = \boxed{-\frac{3}{4}}$ | $x_2^{(2)} = \frac{1}{2}\left(-1 - \frac{1}{2}\right) = \boxed{-3/4}$

(c) $x_1^{(1)} = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$ | $x_1^{(2)} = 0 + 3/4 = \boxed{3/4}$

$x_2^{(2)} = \frac{1}{2}\left(-1 - \frac{1}{2}\right) = \boxed{-\frac{3}{4}}$ | $x_2^{(2)} = \frac{1}{2}\left(-1 - \frac{3}{4}\right) = \boxed{-\frac{7}{8}}$

5. (*6 points*) Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(a) Show that A is positive definite.
(b) Solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (0,1)^T$.
(c) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (0,1)^T$. Taking $\mathbf{x}^(0) = (0,0)^T$ as your initial guess, compute the two iterations of the steepest descent method and obtain the error $\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty$.

(a) symm and $\det(A_1) = \det[2] = 2 > 0 \quad \det(A_2) = \det(A) = 3 > 0$

(b) $$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 3/2 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x_2 = \frac{2}{3} \\ x_1 = \frac{1}{3} \end{array} \quad \boxed{\underset{\sim}{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}}$$

(c) $$r^{(0)} = b - Ax^{(0)} = b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t_0 = \frac{\|r^{(0)}\|_2^2}{\langle r^{(0)}, Ar^{(0)} \rangle} = \frac{1}{2}$$

$$x^{(1)} = x^{(0)} + t_0 r^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}$$

$$r^{(2)} = b - Ax^{(1)} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad t_1 = \frac{\|r^{(1)}\|_2^2}{\langle r^{(1)}, Ar^{(1)} \rangle} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$x^{(2)} = x^{(1)} + t_1 r^{(1)} = \boxed{\begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}}$$

$$\|x^{(2)} - x\|_\infty = \left\| \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ \frac{2}{3} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} -1/12 \\ -1/6 \end{bmatrix} \right\|_\infty = \boxed{\frac{1}{6}}$$

$$\sim 0.\overline{166}$$