

Math 104 A Practice Midterm

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1. Suppose that p^* approximates $p = 900$ with a relative error of at most 10^{-4} . Find the largest interval in which p^* must lie.
2. a) Prove that the Bisection Method converges to the zero (root) of $f(x) = x^2 - 2$ in the interval $[1, 2]$.
b) Find p_3 (and hence an approximation to $p = \sqrt{2}$).
3. A zero of $f(x) = x^2 - 2$ is also a fixed point of $g(x) = x - \frac{1}{4}(x^2 - 2)$.
a) Explain why fixed point iteration using $g(x) = x - \frac{1}{4}(x^2 - 2)$ converges (to $p = \sqrt{2}$) for any p_0 in $[1, 2]$.
b) Compute p_2 starting with $p_0 = 1$.
4. a) Compute p_2 in Newton-Raphson method to find the zero of $f(x) = x^2 - 2$ in the interval $[1, 2]$ beginning with $p_0 = 1$.
b) Which iteration converges the fastest to $p = \sqrt{2}$, fixed point iteration with $g(x) = x - \frac{1}{4}(x^2 - 2)$ or Newton's method? Explain.
5. The following two methods are proposed to compute $5^{1/3}$:

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 5}{3p_{n-1}^2}, \quad (1)$$

$$p_n = \frac{4p_{n-1} + 5/p_{n-1}^2}{5}. \quad (2)$$

Explain, based on the theory seen in class, which method is expected to converge the fastest for a sufficiently good initial guess p_0 .

b) We would like to design a numerical method to solve equation $f(x) = 0$. For this, we consider a fixed point iteration with iterative function

$$g(x) = x - \phi(x)f(x)$$

Determine what the function $\phi(x)$ must be in order to achieve quadratic convergence of the sequence $p_n = g(p_{n-1})$ to the single root p^* .

6. a) Consider the equation $x^2 + \cos x - 10x = 0$ for $x \in [0, 1]$. Show that a solution (zero) of this equation is a fixed point of $g(x) = (x^2 + \cos x)/10$.

- b) Prove that there is a unique fixed point p of g in $[0, 1]$ and hence a unique solution, also p , to $x^2 + \cos x - 10x = 0$ in that interval.
- c) Using $p_0 = 0.15$ as initial guess, obtain p_4 .
- d) Show that this fixed point iteration can only converge linearly to p .
7. Given the function values $f(0) = 1$ and $f(1) = 3$ ($x_0 = 0$, $x_1 = 1$) find the first (linear) Lagrange interpolating polynomial $P_1(x)$ and use it to obtain an approximation to $f(0.5)$.
- b) Find a bound for the error in a) knowing that $|f''(x)| \leq 2$ for all x in $[0, 1]$.
- c) Consider now the function values $f(0) = 1$, $f(1) = 3$, and $f(2) = 11$, find the quadratic interpolating polynomial $P_2(x)$ using Newton's divided difference formula:

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1).$$

Write down a table of the divided differences.

8. Let $x_0 = 0$, $x_1 = 2$, $x_2 = 2.5$, and $x_3 = 3$ and assume we have function values given at these four nodes. Let $x = 1$ be the point corresponding to the 3rd Lagrange polynomial approximation P_{0123} (using the notation in Neville's method). If $P_{0123} = 2$ and $P_{012} = 1.5$, what is P_{123} ?