

# Math 104A Midterm

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Name:

ANSWER KEY

**Instructions:** Write your name clearly. Do all 5 problems. Show all your work.

1. Mark True (T) or False (F) each of the following (no partial credit) statements:

- (a) (~~F~~) A zero of  $f$  is a fixed point of  $g(x) = x^2 - f(x)$ .
- (b) (~~T~~) The error that is produced by representing a number using a finite number of digits is called round-off error.
- (c) (~~T~~) Newton's method is a special form of fixed point iteration for root finding.
- (d) (~~T~~) The Bisection Method always converges to a root or zero of  $f$  in  $[a, b]$  as long as  $f$  is continuous in  $[a, b]$  and  $f(a)f(b) < 0$ .
- (e) (~~F~~) Given  $n + 1$  distinct points  $(x_0, f(x_0)), \dots, (x_n, f(x_n))$ , there is at least one polynomial of degree at most  $n$  that interpolates these points but there could be more than one such polynomial.
- (f) (~~T~~) In Newton's method, the iteration function  $g(x) = x - f(x)/f'(x)$ , satisfies  $g'(p) = 0$ , where  $p$  is a root or zero of  $f$ , and hence it converges at least quadratically when it does converge.

- 2 a) Prove that the Bisection Method converges to the zero (root) of  $f(x) = 2x^2 - 1$  in the interval  $[0, 1]$ .  
 b) Find  $p_3$  (and hence an approximation to  $p = \frac{1}{\sqrt{2}}$ ).  
 c) Find an estimate of the number of iterations  $n$  that would guarantee an error  $|p_n - p|$  to be less than  $10^{-4}$ .

(a)  $f$  is continuous in  $[0, 1]$  and  $f(0)f(1) < 0$   
 $\therefore$  Bisection will converge to a root of  $f$  in  $[0, 1]$ .  
 Since there's only one positive root of  $f$ , bisection will converge to this unique root in  $[0, 1]$

(b)  $p_1 = 0.5$       $f(0.5)f(1) < 0 \Rightarrow p_2 = 0.75$

$f(0.5)f(0.75) < 0 \Rightarrow p_3 = 0.6250$

(c)  $|p_n - p| \leq \frac{1}{2^n} < 10^{-4} \Rightarrow n > \frac{4}{\log_{10}(2)} = 13.28$

$n \geq 14$  iterations

- 3 a) Compute  $p_2$  in Newton's method to find the zero of  $f(x) = x^2 - 2$  in the interval  $[1, 2]$  beginning with  $p_0 = 1$ .  
 b) Which iteration converges the fastest to  $p = \sqrt{2}$ , fixed point iteration with  $g(x) = x - \frac{1}{4}(x^2 - 2)$  or Newton's method? Explain.

$$(a) f'(x) = 2x$$

$$p_{n+1} = p_n - \frac{p_n^2 - 2}{2p_n} \quad n=0, 1, 2, \dots$$

$$\boxed{p_1 = 1.5}, \quad \boxed{p_2 = 1.4166\bar{6}}, \quad \boxed{p_3 = 1.41421568627}$$

(b)  $g'(x) = 1 - \frac{1}{4}2x = 1 - \frac{x}{2}$ ;  $g'(\sqrt{2}) \neq 0$  hence fixed point iteration with this  $g$  can only converge linearly.  $\therefore$  Newton's method converges faster

4. The following two methods are proposed to compute  $5^{1/3}$ :

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 5}{3p_{n-1}^2}, \quad (1)$$

$$p_n = \frac{4p_{n-1} + 5/p_{n-1}^2}{5}. \quad (2)$$

Explain, based on the theory seen in class, which method is expected to converge the fastest for a sufficiently good initial guess  $p_0$ .

b) We would like to design a numerical method to solve equation  $f(x) = 0$ . For this, we consider a fixed point iteration with iterative function

$$g(x) = x - \phi(x)f(x)$$

Determine what the function  $\phi(x)$  must be in order to achieve quadratic convergence of the sequence  $p_n = g(p_{n-1})$  to the single root  $p$ .

(a) For (1)  $g(x) = x - \frac{x^3 - 5}{3x^2}$

(2)  $g(x) = \frac{4x + 5/x^2}{5}$

(1) is Newton's method for  $f(x) = x^3 - 5 = 0$

and thus  $\boxed{g'(5^{1/3}) = 0}$

whereas for (2)  $g'(x) = \frac{4 - 10/x^3}{5}$  and  $g'(5^{1/3}) = \frac{4 - \frac{10}{5}}{5}$

$$g'(5^{1/3}) = \frac{2}{5} \neq 0$$

$\therefore$  (1) will converge the fastest.

(b)  $\boxed{\phi(x) = f'(x)}$  as it'd make  $g'(p) = 0$

5. a) Given the function values  $f(0) = 1$  and  $f(1) = 3$  ( $x_0 = 0$ ,  $x_1 = 1$ ) find the first (linear) Lagrange interpolating polynomial  $P_1(x)$  and use it to obtain an approximation to  $f(0.5)$ .

b) Consider now the function values  $f(0) = 1$ ,  $f(1) = 3$ , and  $f(2) = 11$ , find the quadratic interpolating polynomial  $P_2(x)$ .

$$(a) P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$= \frac{x-1}{0-1} 1 + \frac{x-0}{1-0} 3 = 1-x + 3x$$

$$\boxed{P_1(x) = 2x + 1} \quad f(0.5) \approx \boxed{P_1(0.5) = 2}$$

$$(b) P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-1)(x-2)}{(-1)(-2)} 1 + \frac{(x-0)(x-2)}{(1)(-1)} 3 + \frac{(x-0)(x-1)}{(2)(1)} 11$$

$$= \frac{x^2 - 3x + 2}{2} - 3 \frac{x^2 - 2x}{1} + 11 \frac{x^2 - x}{2}$$

$$= \frac{6x^2 - 2x + 2}{2} = 3x^2 - x + 1$$

$$\boxed{P_2(x) = 3x^2 - x + 1}$$

