A New Approach for the Numerical Solution of Diffusion Equations with Variable and Degenerate Mobility

Hector D. Ceniceros

Department of Mathematics, University of California Santa Barbara, CA 93106

Carlos J. García-Cervera

Department of Mathematics, University of California Santa Barbara, CA 93106

Abstract

We propose a novel approach for the numerical integration of diffusion-type equations with variable and degenerate mobility or diffusion coefficient. Our focus is the Cahn-Hilliard equation which plays a prominent role in phase field models of fluids and soft materials but the methodology has a more general applicability. The central idea is a split method with a linearly implicit component and an analytic step to integrate out the variable mobility. The proposed method is robust, free of high order stability constraints, and its cost is comparable to that of solving the linear Heat Equation with the backward Euler Method. Moreover, by design, the numerical solution is guaranteed to be strictly bounded by the stable, constant states.

Keywords: semi-implicit method, Cahn-Hilliard equation, Allen-Cahn equation, degenerate mobility

1. Introduction

The Cahn-Hilliard equation was introduced [6, 7] as a model for the phase separation that occurs in an isothermal binary system when a spatially uniform mixture is quenched below a critical temperature at which it becomes unstable. Introducing an order parameter ϕ, related to the volume fraction

Email addresses: hdc@math.ucsb.edu (Hector D. Ceniceros), www.math.ucsb.edu/~hdc (Hector D. Ceniceros), cgarcia@math.ucsb.edu (Carlos J. García-Cervera), www.math.ucsb.edu/~cgarcia (Carlos J. García-Cervera)

of the two constituents by $\frac{1}{2}(1+\phi)$ and $\frac{1}{2}(1-\phi)$, the Cahn-Hilliard equation can be written as

$$\frac{\partial \phi}{\partial t} = \nabla \cdot (M(\phi) \nabla \mu(\phi)), \tag{1}$$

where

$$\mu(\phi) = -\epsilon^2 \Delta \phi + f'(\phi) \tag{2}$$

is the chemical potential, which is the first variation of the free energy

$$H[\phi] = \int_\Omega \left[\frac{1}{2} \epsilon^2 |\nabla \phi|^2 + f(\phi) \right] d\mathbf{x}, \tag{3}$$

and $f(\phi(\mathbf{x}))$ is the bulk energy density. The constant ϵ provides a measure of the width of interfacial layers. In most applications, ϵ is taken to be $O(h)$, where h is the mesh size. In (1), $M(\phi)$ is the (concentration-dependent) mobility. To be concrete, we take f to be the symmetric double-well potential:

$$f(\phi) = \frac{1}{4}(1-\phi^2)^2 \tag{4}$$

(thus $\mu = -\epsilon^2 \Delta \phi + \phi^3 - \phi$) and M of the form:

$$M(\phi) = 1 - \gamma^2 \phi^2 \tag{5}$$

with $0 \le \gamma \le 1$. For $\gamma = 1$, bulk diffusion vanishes once the phase domains have settled and the dynamics is then controlled by interfacial diffusion. Because of this, the Cahn-Hilliard equation with degenerate or near degenerate mobility ($\gamma = 1$ or close to 1, respectively) can potentially be a better model for capturing the motion of fluid interfaces when coupled to a flow (Model H [16] or variants of it [18, 4, 22, 24, 23, 25]) as suggested by Zhang and Wang [26]. Elliott and Garcke [13] proved the existence of a weak solution for $\gamma = 1$ but an understanding of the behavior of solutions is still lacking, particularly when $|\phi| \to 1$ and $t \to \infty$. Numerical simulations can shed some light on these important questions.

Phase field models with variable mobility have also been considered in the context of diffusion-induced grain boundary motion [8, 14] and more recently for phase separation when the mobility at the two bulk phases has disparate values [19, 11].

One of the main challenges in the numerical solution of (1)-(2) for $\gamma \approx 1$ and $\epsilon = O(h)$, when h is very small (thin interfaces), is maintaining the

nonnegativity of the mobility $M(\phi)$ and to achieve numerical stability without a prohibitively small time step. Because $\epsilon = O(h)$, for most integration schemes the numerical solution develops over-shoots near an interface and this eventually leads to negative values of M. Naturally, explicit methods are too expensive to be practical due to a severe stability constraint. Some linearly implicit approaches which are efficient for constant mobility [2, 3, 10, 9] become unstable or impractical as $\gamma \to 1$ and ϵ is small. Fully implicit methods have been the predominant approach to deal with a degenerate or near degenerate mobility [26, 21, 15]. But fully implicit schemes have a high cost, can fail to converge as $|\phi| \to \gamma^{-1}$, and do not guarantee that $|\phi| \leq \gamma^{-1}$ for all times. Here, we propose a robust and low cost method that maintains the nonnegativity of M at all times and that allows us to compute the solution with only a linear time-step restriction and without any iteration.

We introduce the method in Section 2 using as motivation a simpler Allen-Cahn-type model and then discuss the corresponding scheme for the Cahn-Hilliard equation along with corresponding numerical results to document its performance.

2. A Semi-Analytic Splitting Method

2.1. Allen-Cahn Type Equations

To motivate the method we consider first equations of the form:

$$\frac{\partial \phi}{\partial t} = M(\phi)F(\phi). \tag{6}$$

Here $F(\phi) = L(\phi) + N(\phi)$ where L is a linear diffusion operator and N is a non-linear, lower order term. We consider only a time discretization and leave space as a continuous variable. Different spatial discretizations and boundary conditions do not represent a limitation of the methodology as it will become clear below.

Consider the following time discretization of equation (6). Given ϕ^n such that $M(\phi^n) > 0$, the approximation at time t_{n+1} is obtained by solving the ordinary differential equation

$$\begin{aligned} \frac{d\varphi}{dt} &= M(\varphi)G^n, \quad t \in (t_n, t_{n+1}], & (7)\\ \varphi(t_n) &= \phi^n, & (8) \end{aligned}$$

where the right hand side, G^n is an approximation to $F(\phi^n) = L(\phi^n)+N(\phi^n)$. We define $\phi^{n+1} = \varphi(t_{n+1})$. The positivity of the mobility follows directly from the following lemma, which is a direct consequence of Picard's theorem:

Lemma 2.1. *Consider the Cauchy problem*

$$\begin{aligned} \frac{du}{dt} &= a[1-\gamma^2 u^2], & (9)\\ u(t_0) &= u_0. & (10) \end{aligned}$$

for given $a \in \mathbb{R}$. If $u_0 \in (-\gamma^{-1}, \gamma^{-1})$, then $u(t) \in (-\gamma^{-1}, \gamma^{-1})$ for all $t \in \mathbb{R}$.

A similar semi-analytic approach has been used in [17, 20] for the Landau-Lifshitz-Gilbert equation in micromagnetics with an explicit time discretization.

For increased stability, we construct G^n treating implicitly the leading order linear term $L(\phi)$ through an intermediate variable ϕ^* as follows:

$$\frac{\phi^* - \phi^n}{\Delta t} = L(\phi^*) + N(\phi^n), \tag{11}$$

or equivalently

$$\phi^* = (I - \Delta t L)^{-1} \left[\phi_n + \Delta t N(\phi_n)\right]. \tag{12}$$

We now define G^n by

$$G^n = \frac{\phi^* - \phi^n}{\Delta t}. \tag{13}$$

Note that

$$G^n = \frac{(I - \Delta t L)^{-1} \left[\phi^n + \Delta t N(\phi^n)\right] - \phi^n}{\Delta t}.$$

and hence

$$G^n = L(\phi^n) + N(\phi^n) + O(\Delta t),$$

which gives us a first order method with complexity comparable to that of solving the linear heat equation using the Backward Euler method.

The method can be understood as a splitting time step:

$$\begin{aligned} \frac{\phi^* - \phi_n}{\Delta t} &= L(\phi^*) + N(\phi_n), & (14)\\ \phi_{n+1} &= u(t_{n+1}), & (15) \end{aligned}$$

where u solves the (diagonal) system of differential equations

$$\frac{du}{dt} = M(u)\left(\frac{\phi^* - \phi^n}{\Delta t}\right) \tag{16}$$

To be specific, let us consider the Allen-Cahn equation:

$$\frac{\partial \phi}{\partial t} = (1 - \gamma\phi^2)\left(\epsilon^2 \Delta\phi + \phi - \phi^3\right). \tag{17}$$

Given $a \in \mathbb{R}$, the solution to the ODE

$$\begin{aligned} \frac{du}{dt} &= a[1 - \gamma^2 u^2], \quad t \in (t_n, t_{n+1}] \\ u(t_n) &= \phi^n \end{aligned}$$

can be written explicitly as

$$u(t;a) = \frac{\frac{e^{2\gamma a(t-t_n)}-1}{\gamma} + \phi^n\left[e^{2\gamma a(t-t_n)} + 1\right]}{\left[e^{2\gamma a(t-t_n)} + 1\right] + \gamma\phi^n\left[e^{2\gamma a(t-t_n)} - 1\right]}. \tag{18}$$

We have written it in this way to make apparent that the solution is well behaved even when $\gamma \to 0$.

We consider the following time-stepping for equation (17). Given ϕ_0 with $|\phi_0| < \gamma^{-1}$:

1. For $n = 1, 2, \ldots,$ solve

$$\frac{\phi^* - \phi^n}{\Delta t} = \epsilon^2 \Delta\phi^* + \phi^n - (\phi^n)^3, \tag{19}$$

   and let

$$G^n = \frac{\phi^* - \phi^n}{\Delta t}. \tag{20}$$

2. Define

$$\phi^{n+1} = u(t_{n+1}; G^n), \tag{21}$$

   where $u(t;a)$ is given by (18).

Note that (19) is an Euler scheme, backward in the diffusion and forward in the nonlinear term, for the original equation with $M \equiv 1$ ($\gamma = 0$). Observe also that (18) can be written as

$$\phi^{n+1} = \frac{1}{\gamma}\,\frac{(1+\gamma\phi^n) - (1-\gamma\phi^n)e^{-2\gamma\Delta t G^n}}{(1+\gamma\phi^n) + (1-\gamma\phi^n)e^{-2\gamma\Delta t G^n}}. \tag{22}$$

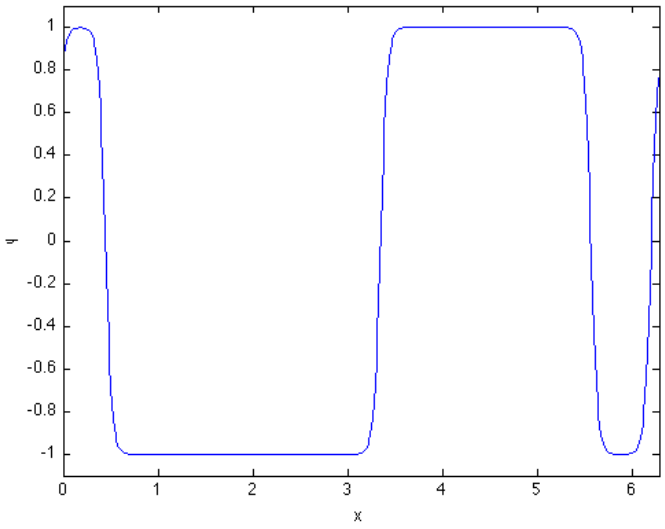

Figure 1: Solution of Ginzburg-Landau type equation at $t = 10000$ computed with the new splitting scheme (19)-(21) with $N = 512$ and $\Delta t = 10$.

We now look at the performance of the method (19)-(21) by considering the 1D initial value problem for the equation (17) with periodic boundary conditions on $[0, 2\pi]$. We discretize in space employing standard second order finite differences. We take $\epsilon = 0.05$, $\gamma = 1$, and use a random initial condition of the form $\phi_j^0 = \epsilon r(j)$, for $j = 0, \ldots, N$ where $r(j) \in U[-1/2, 1/2]$, i.e. it is sampled from the uniform distribution in $[-1/2, 1/2]$. Figure 1 displays the solution at $t = 10000$ computed with $n = 512$ and $\Delta t = 10$. The scheme (19)-(21) behaves as unconditionally stable for all the values of N tested and $|\phi^n| \leq 1$ for all n. As a comparison, the explicit integration of (17) for $\epsilon = 0.05$ and $N = 524$ requires $\Delta t \leq 0.01$. While the time-step and the integration are still manageable using explicit methods in this case (due to the small ϵ), numerical overshoots will eventually lead to a numerical solution for which $\max |\phi^n| > 1$. This does not occur with the new method (19)-(21) for which $\max |\phi^n| \leq 1$ for all n. Moreover, the situation with degenerate mobility becomes intractable with explicit methods for Cahn-Hilliard type equations.

2.2. The Cahn-Hilliard Equation

We now look at the Cahn-Hilliard equation with variable mobility as given by equations (1)-(2) and (4)-(5). We rewrite the equation as

$$\frac{\partial \phi}{\partial t} = M(\phi)\left[\Delta\mu(\phi) + \nabla(\log M(\phi)) \cdot \nabla\mu(\phi)\right] \tag{23}$$

to apply the splitting strategy used for the Allen-Cahn type equation. As before, we extract linear leading order terms to be treated *implicitly* in the first split step and write the term in brackets in (23) as

$$\Delta\mu(\phi) + \nabla(\log M(\phi)) \cdot \nabla\mu(\phi) = -\epsilon^2\Delta^2\phi + \alpha_s\Delta\phi + F(\phi), \tag{24}$$

where

$$F(\phi) = \Delta\left(\phi^3 - \phi\right) - \alpha_s\Delta\phi + \nabla(\log M(\phi)) \cdot \nabla\mu(\phi). \tag{25}$$

Here α_s is a scalar and the linear term $\alpha_s\Delta\phi$ arises from $\Delta\left(\phi^3 - \phi\right)$ by observing that $\Delta\left(\phi^3 - \phi\right) = \nabla\cdot\nabla\left(\phi^3 - \phi\right) = \nabla\left(3\phi^2 - 1\right)\nabla\phi$. It was proved in [12] that for the variable coefficient diffusion equation $u_t = \nabla\cdot(a(u)\nabla u)$ the following discretization

$$\frac{u^{n+1} - u^n}{\Delta t} = \alpha_s\nabla u^{n+1} + \nabla\cdot(a(u^n)\nabla u^n) - \alpha_s\nabla u^n \tag{26}$$

is unconditionally stable for $\alpha_s \geq \frac{1}{2}\|a(u)\|_\infty$. This type of linearized semi-implicit splitting was employed successfully in the Cahn-Hilliard equation context and for other phase field models in [2, 3, 10, 9]. It has also been used with variable mobility but in this case it requires a quadratic stability constraint and does not guarantee that $|\phi^n| \leq \gamma^{-1}$ for all n (see e.g. [2]).

A first semi-analytic splitting method for the Cahn-Hilliard equation with variable mobility can be implemented as follows: Given ϕ_0 with $|\phi_0| < \gamma^{-1}$:

1. For $n = 1, 2, \ldots,$ solve

$$\frac{\phi^* - \phi^n}{\Delta t} = -\epsilon^2\Delta^2\phi^* + \alpha_s\Delta\phi^* + F(\phi^n), \tag{27}$$

   Let

$$G^n = \frac{\phi^* - \phi^n}{\Delta t}. \tag{28}$$

2. Define

$$\phi^{n+1} = u(t_{n+1}; G^n), \tag{29}$$

   where $u(t; a)$ is given by (22).

As a test for the method we consider a case of phase separation in $[0, 2\pi] \times [0, 2\pi]$ with the interfacial thickness parameter $\epsilon = 2h$ where $h = \Delta x = \Delta y = 2\pi/N$ and periodic boundary conditions. We use standard second order finite differences in space. The initial condition is a random perturbation from a homogeneous equilibrium state $\bar{\phi}_0$, namely

$$\phi_0(x, y) = \bar{\phi}_0 + \epsilon\, \xi \qquad \text{for } (x, y) \in [0, 2\pi] \times [0, 2\pi], \tag{30}$$

where $\xi \in U[-1/2, 1/2]$ and $\epsilon = 2h$ as in the chemical potential (2).

The scheme (27)-(29) works very well for γ up to about 0.95. It behaved as nearly unconditionally stable in our tests. As an illustration, for $\gamma = 0.95$, starting with the initial condition (30) with $\bar{\phi}_0 = 0$ and $N = 256$ we find that a Δt as high as 0.1 is sufficient to maintain stability and capture the phase separation process. Figure 2 displays the solution ϕ at different times up to $T = 10000$. For this computation $N = 256$, $\Delta t = 0.1$, and $\alpha_s = 1$. During the entire computation the energy (3) decreases monotonically as Fig. 3 demonstrates. The coarsening dynamics is rather slow due to the small bulk diffusion; it takes an enormous time, $t = 10000$, to reach near a potential stationary state. Interestingly, note that a circular interface appears to have been selected instead of the usual stripes in a spinodal decomposition with constant mobility (bulk dominated diffusion) starting from a perturbation of the equal composition state $\bar{\phi}_0 = 0$.

The degenerate case $\gamma = 1$ is computationally more challenging. Unless the interfaces are very wide ($\epsilon >> O(h)$), when $\gamma = 1$ the method (27)-(29) only allow us to compute the solution for relatively short times and it is not robust as the spatial resolution increases (and ϵ decreases). Even though by construction the semi-analytic method produces $|\phi_n| \leq 1$ for all n, round off errors can prematurely lead to a zero mobility in the log evaluation and consequently a break down in the computation. A fully implicit method shares a similar problem; as the solution ϕ approaches ± 1 the associated linear system becomes highly ill-conditioned and can become singular due to numerical error.

To overcome the aforementioned problem for $\gamma = 1$, we introduce two modifications to the method (27)-(29). First, the logarithmic term in (25) is approximated by

$$\log M(\phi) \approx \log \left(|M(\phi)| + \epsilon^2 \right) \equiv \log^{\epsilon} M(\phi) \tag{31}$$

where $\epsilon = 2h$ so that we retain second order accuracy in space. Second, we substitute the backward-forward Euler scheme to solve for ϕ^* with an

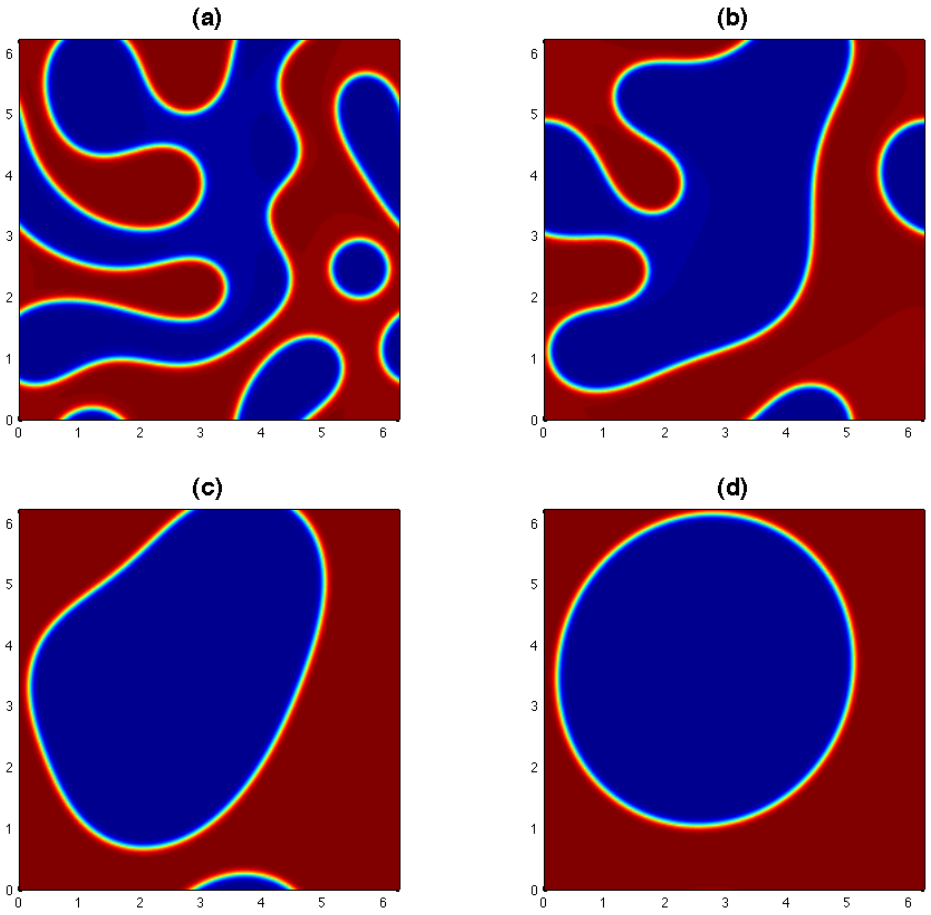

Figure 2: Flooded contour plots of ϕ for $\gamma = 0.95$, $N = 256$, $\Delta t = 0.1$, and $\epsilon = 2(2\pi/N)$. (a) $t = 200$, (b) $t = 1000$, (c) $t = 2000$, and (d) $t = 10000$.

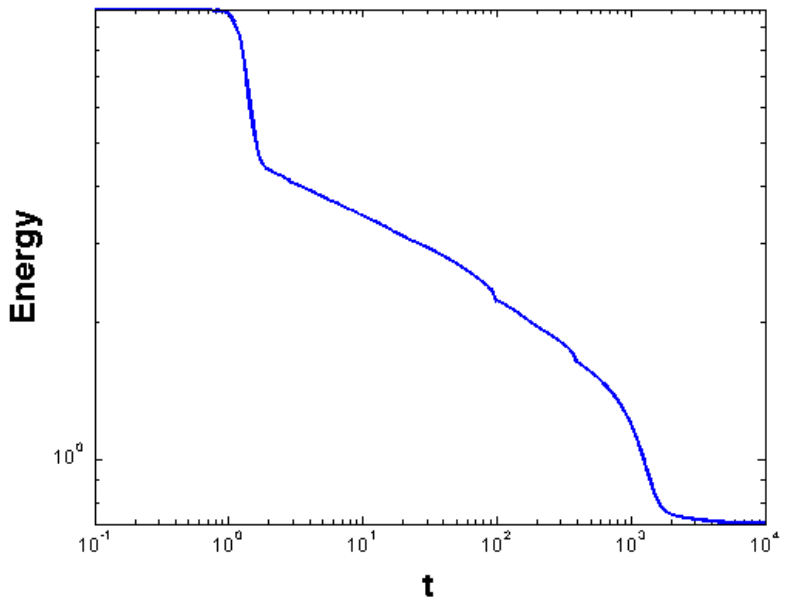

Figure 3: Energy vs time (using logarithmic scales) for $\gamma = 0.95$, $N = 256$, $\Delta t = 0.1$

L-stable, implicit-explicit Runge Kutta (IMEX RK) method to provide sufficient damping [1]. Specifically, consider the ODE system

$$\frac{d\varphi}{dt} = G(\varphi) + F(\varphi), \tag{32}$$

where G and F correspond to (the discretrized versions of)

$$G(\varphi) = -\epsilon^2 \Delta^2 \varphi + \alpha_s \Delta \varphi, \tag{33}$$

$$F(\phi) = \Delta\left(\varphi^3 - \varphi\right) - \alpha_s \Delta \varphi + \nabla(\log^\epsilon M(\varphi)) \cdot \nabla \mu(\varphi). \tag{34}$$

Then, the diagonally implicit Runge Kutta (DIRK) method that we propose to solve for ϕ^* is a two-stage explicit, two stage implicit, and second order method which can be described as follows. Let $\beta = (2 - \sqrt{2})/2$ and $\delta = 1 - 1/(2\beta)$.

1. Let $\hat{K}_1 = F(\phi_n)$.
2. Solve $\phi_1 = \phi_n + \Delta t \beta G(\phi_1) + \Delta t \beta \hat{K}_1$.
3. Let $K_1 = G(\phi_1)$ and $\hat{K}_2 = F(\phi_1)$.
4. Solve $\phi^* = \phi_n + \Delta t(1-\beta) K_1 + \Delta t \beta G(\phi^*) + \Delta t \delta \hat{K}_1 + \Delta t(1-\delta)\hat{K}_2$.

We tested several of the first and second order IMEX RK methods in [1] and found that this scheme provided the best performance in terms of cost and stability. Following [1] we refer to it as DIRK222 and write symbolically

$$\phi^* = \text{DIRK222}(\phi^n). \tag{35}$$

The modified semi-analytic method suitable for all values of γ in $(0, 1]$ reads: Given ϕ_0 with $|\phi_0| < \gamma^{-1}$:

1. For $n = 1, 2, \ldots,$ solve

$$\phi^* = \text{DIRK222}(\phi^n) \tag{36}$$

   Let

$$G^n = \frac{\phi^* - \phi^n}{\Delta t}. \tag{37}$$

2. Define

$$\phi^{n+1} = u(t_{n+1}; G^n), \tag{38}$$

   where $u(t; a)$ is given by (22).

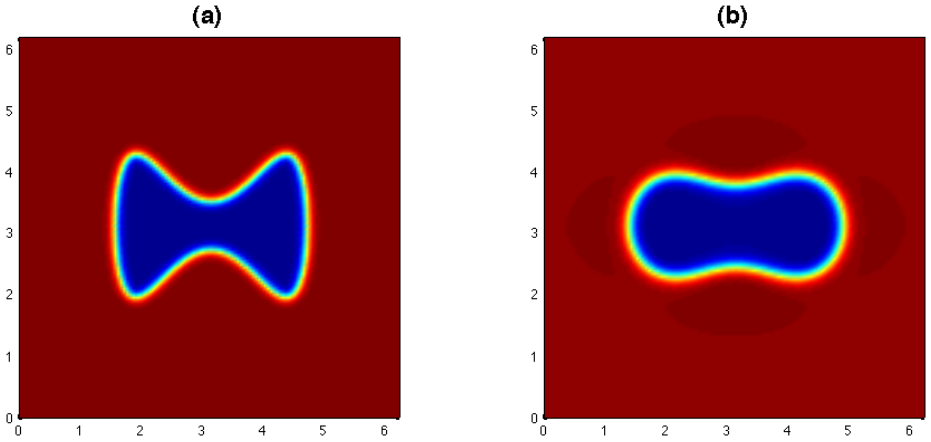

Figure 4: (a) Initial condition (39) for $\epsilon = 2(2\pi/N)$, $N = 128$ and (b) ϕ at $T = 5$ ($\Delta t = 0.01/16$).

We test the method (36)-(38) for $\gamma = 1$ for two sets of initial conditions which are common in applications of the phase field models. The random initial condition (30) and a deterministic one consisting of a dumbbell shape domain given by

$$\phi_0(x, y) = A \tanh\left(\frac{d(x, y)}{\epsilon}\right), \tag{39}$$

where d is the signed distance to the curve C expressed in parametric form by

$$x_c(\alpha) = a\cos(\alpha) + \pi, \tag{40}$$

$$y_c(\alpha) = \sin(\alpha) + b\sin(3\alpha) + \pi, \tag{41}$$

for $\alpha \in [0, 2\pi]$. We take $A = 0.95$, $a = \pi/2$, and $b = 0.60$. This initial ϕ is displayed in a flooded contour plot in Fig. 4(a). Note that the interface has a very high curvature and a thickness of $O(\epsilon)$. We begin with this deterministic initial condition to check the order of convergence and stability of the method for $\gamma = 1$. We take $N = 128$ and vary Δt accordingly to compute the ratio

$$q_{\Delta t} = \frac{|\phi_{\Delta t} - \phi_{\Delta t/2}|}{|\phi_{\Delta t/2} - \phi_{\Delta t/4}|}, \tag{42}$$

where $\phi_{\Delta t}$ stands for the numerical approximation of the solution computed with a given Δt up to time $T = 5$. Table 1 shows $q_{\Delta t}$ for $\Delta t = 0.01$, 0.005,

Table 1: Convergence ratio $q_{\Delta t}$ for $\gamma = 1$, $N = 128$.

Δt	0.010	0.005	0.0025
$q_{\Delta t}$	2.48	2.15	3.72

and 0.0025. The theoretical first order convergence rate in time is clearly verified by these numbers. The method is still stable for Δt as large as 0.05 but of course is not very accurate for such a large Δt.

We now compute the solution again for initial condition (39) but with $A = 0.90$ and an even thinner interface, $\epsilon = 2(2\pi/N)$ with $N = 512$. The maximum value of $|\phi|$ initially is about 0.9025. We take $\Delta t = 0.00025$. This choice of Δt is due to accuracy concerns and not stability; the method remains stable with a Δt at least four times bigger.

The system evolves in two stages. First, there is both bulk and interfacial dynamics during which $|\phi| \to 1$ outside the narrow interfacial region. This process can be seen in Fig. 6(b), corresponding to $t = 20$, where the bulk is not completely uniform. By about $t = 50$, $|\phi| = 1$ to machine precision in the entire bulk and a slower interfacial dynamics begins. According to the asymptotics of Cahn, Elliot, and Novick-Cohen [5], this interfacial dynamics corresponds to motion by the surface Laplacian of the mean curvature. Figure 6(c)(d) shows the interfacial changes after a long time ($t = 100$ and $t = 200$).

While the free energy decreases monotonically during the entire computation (Fig. 6), there was a variation of $\bar{\phi}$, the mean value of ϕ. This lack of preservation of $\bar{\phi}$ is not surprising as the method is not in conservation form. At the end of the long time computation, after 8×10^5 time-steps (corresponding to $t = 200$), we observed a relative change in $\bar{\phi}$ of about 2%. For $t = O(10)$, that variation was less than 0.2%.

One possibility for achieving conservation of $\bar{\phi}$ to machine precision (approximately 10^{-16}) is to use a predictor-corrector approach as follows. Let $\tilde{\phi}^{n+1}$ denote the numerical solution at t_{n+1} computed with the proposed method (predictor). Then, the corrector step is defined as

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \nabla_h \cdot \left(M(\tilde{\phi}^{n+1}) \nabla_h \tilde{\phi}^{n+1} \right). \tag{43}$$

We tested this scheme with the same initial condition for various N and Δt to examine its stability. We observed that (43) does preserve $\bar{\phi}$ to machine

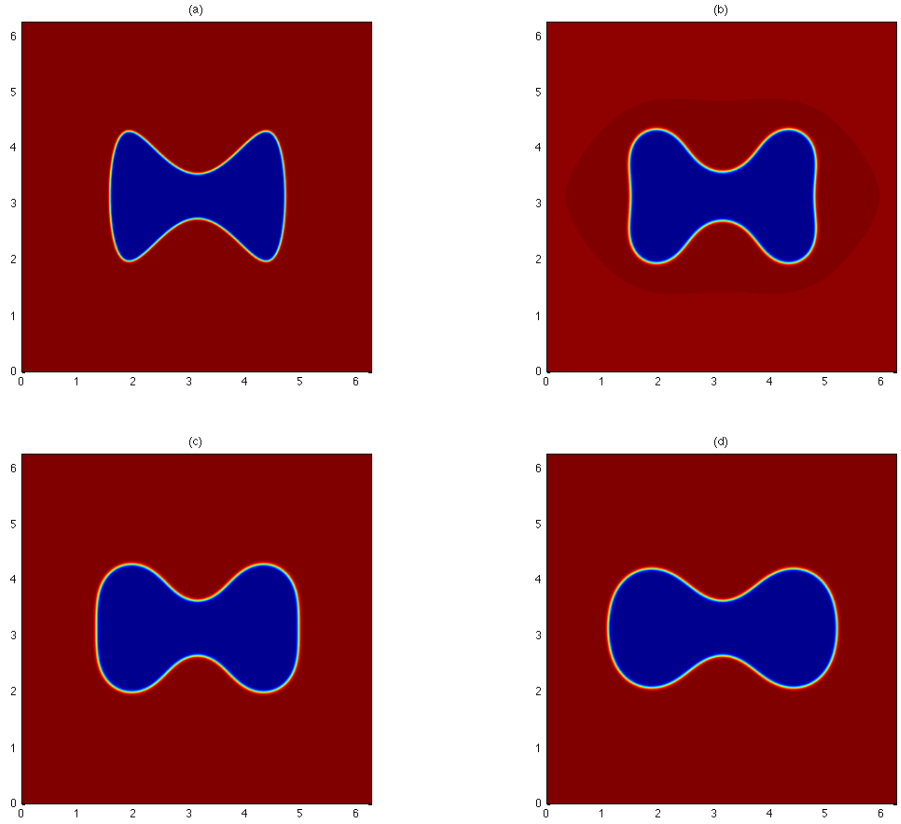

Figure 5: Flooded contour plots of ϕ for initial condition (39) with $A = 0.90$ and $\gamma = 1$. $N = 512$, $\Delta t = 0.00025$, and $\epsilon = 2(2\pi/N)$. (a) $t = 0$, (b) $t = 20$, (c) $t = 100$, and (d) $t = 200$.

precision for all times but it requires a smaller Δt and a quadratic stability constraint. For example, for $N = 256$ the predictor-corrector scheme (43) is stable with $\Delta t = 1.25 \times 10^{-4}$. We note, however, that the numerical solution of (43) can no longer be guaranteed to satisfy $|\phi^n| \leq 1$ for all n. Indeed, our numerical simulations confirmed that this property is lost.

The DIRK based semi-analytic method (36)-(38) also works well for the degenerate case ($\gamma = 1$) with random initial conditions. We now take $\bar{\phi}_0 = 0.4$ in (30) which favors the positive phase and leads to the nucleation of the negative phase. The numerical parameters are $N = 256$, $\Delta t = 0.001$, $\epsilon = 2(2\pi/N)$, and $\alpha_s = 1$. We note that an order of magnitude larger Δt is sufficient for stability but to appropriately capture the nucleation and

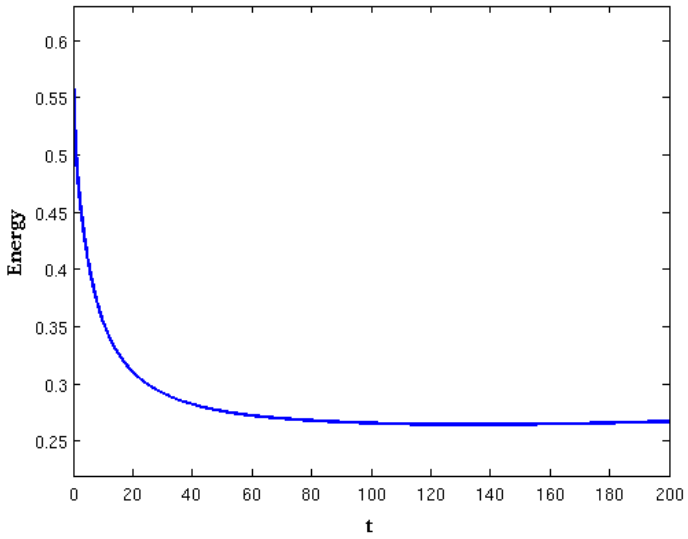

Figure 6: Energy vs time for $\gamma = 1$, $N = 512$, $\Delta t = 0.00025$. Initial condition (39).

coarsening the smaller Δt was necessary. Figure 7 shows the nucleation process and the slow coarsening dynamics.

Finally, we note that due to the low cost of the proposed method (comparable to that of explicit Euler per time step) and its efficiency (free of high order stability constraints), we were able to do all computations, including the very long time, steady state computations with a modest laptop computer in the course of a few hours.

3. Conclusions

We presented a new general strategy for the computation of diffusion problems with variable and degenerate mobility. The approach is based on a semi-analytic splitting to integrate out exactly the mobility and to discretize implicitly leading order terms. We demonstrated with the Cahn-Hilliard equation that this strategy can produce stable and robust schemes even in the near degenerate and degenerate cases. We believe that the proposed methodology could be a powerful tool to investigate the behavior of solution and potential singularity to degenerate phase field models when $\phi \to 1$.

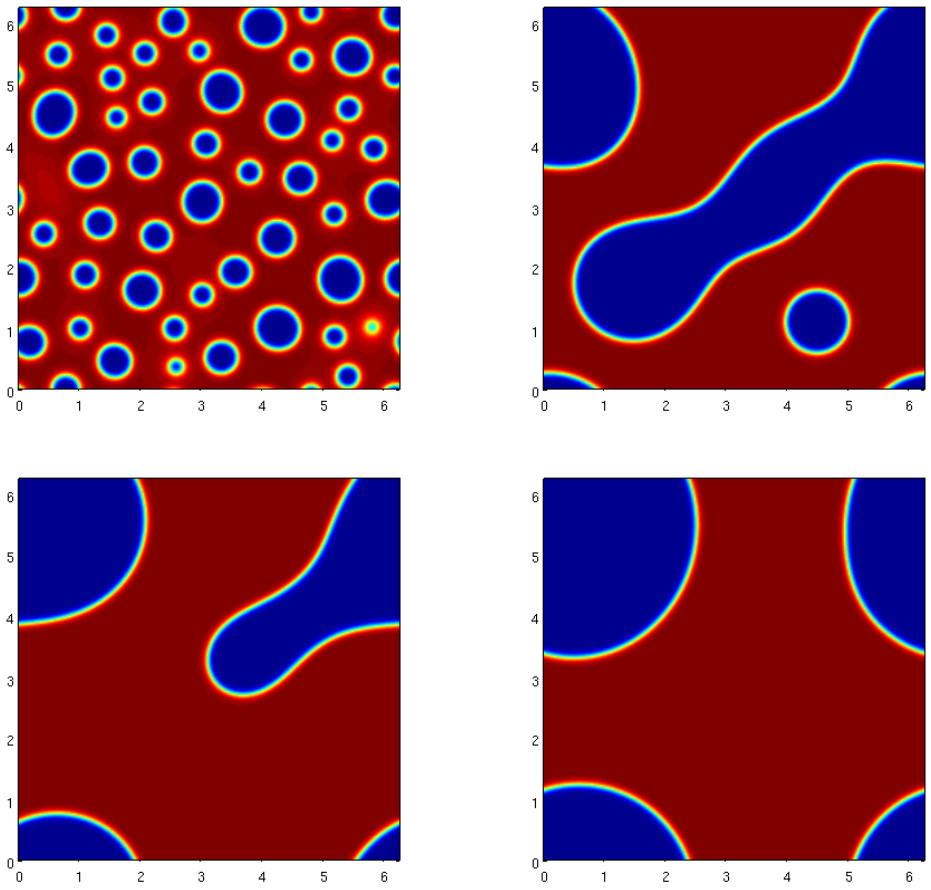

Figure 7: Flooded contour plots of ϕ for initial condition (30) with $\bar{\phi}_0 = 0.4$ and $\gamma = 1$. $N = 512$, $\Delta t = 0.001$, and $\epsilon = 2(2\pi/N)$. (a) $t = 5$, (b) $t = 500$, (c) $t = 2000$, and (d) $t = 4000$.

Acknowledgments

We gratefully acknowledge partial support for this work by the National Science Foundation under grants DMS 1016310 and DMR 1035480. We would like to thank Prof. Alexandre Roma for communicating to us the difficulties associated with the computation of near degenerate mobility in the Cahn-Hilliard equation. We also thank Prof. Juan J. Tapia Armenta for his suggestions which helped us improve the presentation of our manuscript.

References

[1] U. M. Ascher, S. J. Ruuth, and R. J. Spiteri. Implicit-explicit Runge-Kutta methods for time dependent partial differential equations. *Applied Numerical Mathematics*, 25:151–167, 1997.

[2] V. E. Badalassi, H. D. Ceniceros, and S. Banerjee. Computation of multiphase systems with phase field models. *J. Comput. Phys.*, 190:371–397, 2003.

[3] V. E. Badalassi, H. D. Ceniceros, and S. Banerjee. Gravitational effects on structural development in quenched complex fluids. *Ann. NY. Acad. Sci.*, 1027:371–382, 2004.

[4] F. Boyer. A theoretical and numerical model for the study of incompressible mixture flows. *Computer & Fluids*, 31(1):41, 2002.

[5] J. W. Cahn, C.M. Elliott, and A. Novick-Cohen. The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature . *Euro. J. Appl. Math.*, 7:287–301, 1996.

[6] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system I. *J. Chem. Phys.*, 28:258, 1958.

[7] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system III. *J. Chem. Phys.*, 31:688, 1959.

[8] J.W. Cahn, P Fife, and O Penrose. A phase-field model for diffusion-induced grain-boundary motion. *Acta Mater.*, 45(10):4397–4413, 1997.

[9] H. D. Ceniceros, R. L. Nos, and A. M. Roma. Three-dimensional, fully adaptive simulations of phase-field fluid models. *J. Comput. Phys.*, 229(17):6135–6155, 2010.

[10] H. D. Ceniceros and A. M. Roma. A nonstiff, adaptive, mesh refinement-based method for the Cahn-Hilliard equation. *J. Comput. Phys.*, 225(2):1849–1862, 2007.

[11] S. Dai and Q. Du. Motion of interfaces governed by the Cahn-Hilliard equation with highly disparate diffusion mobility. *SIAM J. Appl. Math.*, 72(6):1818–1841, 2012.

[12] J. J. Douglas and T. Dupont. Alternating-direction galerkin methods on rectangles. In B. Hubbard, editor, *SYNSPADE-1970, Numerical Solution of Partial Differential Equations-II*, pages 133–213, New York, 1971. Academic Press.

[13] C. M. Elliott and H. Garcke. On the Cahn-Hilliard equation with degenerate mobility. *SIAM J. math. Anal.*, 27(2):404–423, 1996.

[14] P.C. Fife, J. W. Cahn, and C.M. Elliott. A free boundary problem for diffusion induced grain boundary motion . *Interfaces and Free Boundaries*, 3:291–336, 2001.

[15] H. Gomez and T. J. R. Hughes. Provably unconditionally stable, second-order time-accurate, mixed variational methods for phase-field models. *J. Comput. Phys.*, 230(13):5310–5327, 2011.

[16] P. C. Hohenberg and B. I. Halperin. Theory of dynamic critical phenomena. *Rev. Mod. Phys.*, 49(3):435, 1977.

[17] J. S. Jiang, H. G. Kaper, and G. K. Leaf. Hysteresis in layered spring magnets. *Discrete and Continuous Dynamical Systems-Series B*, 1(2):219–232, 2001.

[18] J. Lowengrub and L. Truskinovsky. Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *Proc. R. Soc. London A*, 454:2617, 1998.

[19] G. Sheng, T. Wang, Q. Du, K. G. Wang, Z. K. Liu, and L. Q. Chen. Coarsening kinetics of a two phase mixture with highly disparate diffusion mobility. *Commun. Comput. Phys.*, 8(2):249–264, 2010.

[20] B. Van de Wiele, F. Olyslager, and L. Dupre. Fast semianalytical time integration schemes for the Landau-Lifshitz equation. *Magnetics, IEEE Transactions on*, 43(6):2917 –2919, june 2007.

[21] O. Wodo and B. Ganapathysubramanian. Computationally efficient solution to the Cahn-Hilliard equation: Adaptive implicit time schemes, mesh sensitivity analysis and the 3D isoperimetric problem. *J. Comput. Phys.*, 230(15):6037–6060, 2011.

[22] P. Yue, J. J. Feng, C. Liu, and J. Shen. A diffuse-interface method for simulating two-phase flows of complex fluids. *J. Fluid Mech.*, 515:293–317, 2004.

[23] P. Yue, J. J. Feng, C. Liu, and J. Shen. Diffuse-interface simulations of drop-coalescence and retraction in viscoelastic fluids. *J. non-Newtonian Fluid Dynamics*, 129:163–176, 2005.

[24] P. Yue, J. J. Feng, C. Liu, and J. Shen. Viscoelastic effects on drop deformation in steady shear. *J. Fluid Mech.*, 540:427–437, 2005.

[25] P. Yue, C. Zhou, J. J. Feng, C. F. Ollivier-Gooch, and H. H. Hu. Phase-field simulations of interfacial dynamics in viscoelastic fluids using finite elements with adaptive meshing. *J. Comput. Phys.*, 219:47–67, 2006.

[26] T. Zhang and Q. Wang. Cahn-Hilliard vs singular Cahn-Hilliard equations in phase field modeling. *Commun. Comput. Phys.*, 7(2, SI):362–382, 2010.