18 Finite differences for the wave equation

Similar to the numerical schemes for the heat equation, we can use approximation of derivatives by difference quotients to arrive at a numerical scheme for the wave equation $u_{tt} = c^2 u_{xx}$. Since both time and space derivatives are of second order, we use centered differences to approximate them. Taking a mesh of size Δt in the t variable, and a mesh of size Δx in the x variable, we have at the grid points $(x_j, t_n) = (j\Delta x, n\Delta t)$,

$$\begin{aligned} u_{tt}(j\Delta x, n\Delta t) &= \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} + \mathcal{O}(\Delta t)^2, \\ u_{xx}(j\Delta x, n\Delta t) &= \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \mathcal{O}(\Delta x)^2, \end{aligned}$$

where we used the notation $u_j^n = u(j\Delta x, n\Delta t)$, for $j = 0, 2, \dots J$, $n = 0, 1, \dots, N$. Then up to an error of order $\mathcal{O}((\Delta x)^2 + (\Delta t)^2)$ the wave equation at the lattice points can be replaced by the following difference equation

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}. \tag{1}$$

Denoting

$$s = c^2 \frac{(\Delta t)^2}{(\Delta x)^2},$$

and solving for u_j^{n+1}, we can rewrite (1) as

$$u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + 2(1-s)u_j^n - u_j^{n-1}. \tag{2}$$

This is an explicit scheme for finding the numerical solution $\{u_j^n\}$. Notice that to find u_j^{n+1} the scheme (2) uses values at two previous time steps. Schematically this is given by the following diagram

$$\begin{array}{ccc} & * & \\ s\bullet & (2-2s)\bullet & s\bullet \\ & -1\bullet & \end{array}$$

As we saw in the case of the explicit FTCS scheme for the heat equation, the value of s has a crucial effect on the stability of the numerical scheme. Let us consider a few values for the parameter s.

$\underline{\mathbf{s = 2}}$. With this choice of s the scheme becomes

$$u_j^{n+1} = 2(u_{j+1}^n - u_j^n + u_{j-1}^n) - u_j^{n-1}. \tag{3}$$

To kick start this marching scheme, we need the values at two initial time steps, for which we take

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \end{array} \tag{4}$$

Recall that the exact solution of the wave equation from an initial box wave separates into two equal waves traveling in opposite directions. We expect a similar behavior from the approximated solution.

Using the initial values (4) we march forward in time with the scheme (3). The template for the scheme and the numerical solution are given below.

$$\begin{array}{ccc} & * & \\ 2\bullet & -2\bullet & 2\bullet \\ & -1\bullet & \end{array}$$

8	−12	4	13	−22	13	4	−12	8
0	4	−2	−3	6	−3	−2	4	0
0	0	2	1	−2	1	2	0	0
0	0	0	1	2	1	0	0	0
0	0	0	1	2	1	0	0	0

As we can see the scheme leads to unexpected large values, and hence is unstable.

<u>**s = 1**</u>. With this value of s the scheme becomes

$$u_j^{n+1} = u_{j+1}^n + u_{j-1}^n - u_j^{n-1}. \tag{5}$$

Using the same two initial steps (4), we arrive at the following numerical solution using the template below.

$$\begin{array}{ccc} & * & \\ 1\bullet & & 1\bullet \\ & -1\bullet & \end{array}$$

1	1	0	0	0	0	0	1	1
0	1	1	0	0	0	1	1	0
0	0	1	1	0	1	1	0	0
0	0	0	1	2	1	0	0	0
0	0	0	1	2	1	0	0	0

We can clearly see the two separated waves traveling in opposite directions, hence this value of s leads to an approximate solution that mimics the behavior or the exact solution.

Let us now analyze the stability of the scheme (2) by plugging in the separated solution $u_j^n = \xi^n e^{ikj\Delta x}$. The stability condition is $|\xi| \leq 1$, since otherwise the scheme would lead to exponentially growing solutions.

$$\xi^{n+1} e^{ikj\Delta x} = s(\xi^n e^{ikj\Delta x} + \xi^n e^{-ikj\Delta x}) + 2(1-s)\xi^n e^{ikj\Delta x} - \xi^{n-1} e^{ikj\Delta x}.$$

Dividing both sides of the above identity by u_j^n, and collecting the remaining ξ terms on the left, we get

$$\xi + \frac{1}{\xi} = s(e^{ik\Delta x} + e^{-ik\Delta x}) + 2(1-s) = 2 + 2s(\cos k\Delta x - 1).$$

Denoting the right hand side by $p = 2 + 2s(\cos k\Delta x - 1)$, we observe that $p \leq 2$, since $\cos k\Delta x - 1 \leq 0$. Then we have the equation

$$\xi + \frac{1}{\xi} = p, \quad \text{or} \quad \xi^2 - p\xi + 1 = 0.$$

The roots of the last quadratic equation are

$$\xi_\pm = \frac{p \pm \sqrt{p^2 - 4}}{2}.$$

Now, if the discriminant is positive, i.e. $p^2 - 4 > 0$, which implies that $p < -2$, since p cannot be larger than 2, then the quadratic equation will have two distinct real roots. But one of these roots is

$$\xi_- = \frac{p - \sqrt{p^2 - 4}}{2} < \frac{p}{2} < \frac{-2}{2} = -1,$$

which would lead to instability.

If, on the other hand, the discriminant is nonpositive, i.e. $p^2 - 4 \leq 0$, which implies $-2 \leq p \leq 2$, then the roots of the quadratic equation are complex,

$$\xi_\pm = \frac{p}{2} \pm i\frac{\sqrt{4 - p^2}}{2}, \quad \text{with norm} \quad |\xi_\pm| = \sqrt{\frac{p^2}{4} + \frac{4 - p^2}{4}} = \sqrt{1} = 1.$$

This is consistent with our intuition about solutions of the wave equation, since the time component of the separated solution will be

$$T_n = \xi^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

So the stability condition is equivalent to the requirement that the discriminant is nonpositive, $p^2 - 4 \leq 0$, or $-2 \leq p \leq 2$. Substituting the expressions of p in terms of s, we have

$$-2 \leq 2 + 2s(\cos k\Delta x - 1) \leq 2.$$

The inequality on the right is always satisfied, while the worst case for the left inequality is when $\cos k\Delta x \approx -1$, which leads to the inequality $-2 \leq 2 - 4s$, so the stability condition is

$$s = c^2 \frac{(\Delta t)^2}{(\Delta x)^2} \leq 1. \tag{6}$$

Notice that if we define the speed of the numerical scheme to be $\Delta x/\Delta t$, then the stability condition (6) implies that

$$\text{speed of the scheme} \geq c,$$

with c being the speed of the exact solution. Thus, the necessary condition for stability of the numerical scheme is that the speed of the scheme must be at least as large as the speed of the exact equation.

Another way of understanding this stability is to compare the domains of dependence of the exact equation and the numerical scheme. We can see from (2) that the interval of dependence of u_j^n is

$$[(j-n)\Delta x, (j+n)\Delta x], \tag{7}$$

while the interval of dependence for the exact solution at the same point $(j\Delta x, n\Delta t)$ is

$$[j\Delta x - cn\Delta t, j\Delta x + cn\Delta t]. \tag{8}$$

Using the stability condition $c\Delta t \leq \Delta x$, we compare (7) and (8) and observe that

$$[j\Delta x - cn\Delta t, j\Delta x + cn\Delta t] \subset [(j-n)\Delta x, (j+n)\Delta x].$$

Hence, the stability condition is equivalent to requiring that the domain of dependence of the scheme contain the domain of dependence of the exact solution. This is called the CFL condition, first formulated by Courant, Friedrichs and Lewy for general hyperbolic equations.

18.1 Initial conditions

We saw that to kick-start the scheme (2) one needs values at the two initial time steps. The initial conditions for the wave equation, however, have the form

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x). \tag{9}$$

The value at time step $n = 0$ can be found from the first initial condition, $u_j^0 = \phi_j = \phi(j\Delta x)$, while the second condition will be approximated by a difference quotient. The backward and forward difference approximations will introduce an error of order $\mathcal{O}(\Delta x)$, which is larger than the truncation error of the difference equation that has order $\mathcal{O}(\Delta x)^2$. Thus, we use the centered difference to write

$$\frac{u_j^1 - u_j^{-1}}{2\Delta t} = \psi_j = \psi(j\Delta x), \tag{10}$$

where u_j^{-1} are ghost points, similar to those used for resolving the Neumann boundary conditions.

For $n = 0$ the scheme (2) gives

$$u_j^1 + u_j^{-1} = s(u_{j+1}^0 + u_{j-1}^0) + 2(1-s)u_j^0,$$

which along with (10), and the identification $u_j^0 = \phi_j$ will lead to the system

$$\begin{cases} u_j^1 + u_j^{-1} = s(\phi_{j+1} + \phi_{j-1}) + 2(1-s)\phi_j \\ u_j^1 - u_j^{-1} = 2\Delta t \psi_j. \end{cases}$$

We can obtain the values u_j^1 at the time step $n = 1$ from this system by adding the equations together. Along with the first initial condition, this will give the values at two initial time steps in terms of the initial data (ϕ, ψ),

$$\begin{aligned} u_j^0 &= \phi_j, \\ u_j^1 &= \frac{s}{2}(\phi_{j+1} + \phi_{j-1}) + (1-s)\phi_j + \Delta t \psi_j. \end{aligned}$$

Using these, one can march forward in time using the scheme (2) to find the numeric solution.

18.2 Conclusion

Using the centered differences for the second order time and space derivatives we derived a difference equation equivalent to the wave equation up to an error of order $\mathcal{O}(\Delta x)^2$, which lead to an explicit numerical scheme. Analyzing the stability of this scheme we arrived at the condition $c\Delta t \leq \Delta x$, where c is the wave speed. This condition means that the wave speed of the numeric scheme must be at least as large as the wave speed of the exact equation.

We also observed that the derived scheme uses two previous time steps to compute the values of the numerical solution at a particular grid point, thus one needs the values of two initial times steps to run the scheme. These we were able to find from the initial conditions by centered difference approximation, which does not add smaller order errors to the numerical scheme. Boundary conditions for the wave equation can be handled precisely as in the case of the heat equation discussed last time.