THE CLOSED-POINT ZARISKI TOPOLOGY
FOR IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. In previous work, the second author introduced a topology, for spaces of irreducible representations, that reduces to the classical Zariski topology over commutative rings but provides a proper refinement in various noncommutative settings. In this paper, a concise and elementary description of this refined Zariski topology is presented, under certain hypotheses, for the space of simple left modules over a ring R. Namely, if R is left noetherian (or satisfies the ascending chain condition for semiprimitive ideals), and if R is either a countable dimensional algebra (over a field) or a ring whose (Gabriel-Rentschler) Krull dimension is a countable ordinal, then each closed set of the refined Zariski topology is the union of a finite set with a Zariski closed set. The approach requires certain auxiliary results guaranteeing embeddings of factor rings into direct products of simple modules. Analysis of these embeddings mimics earlier work of the first author and Zimmermann-Huisgen on products of torsion modules.

1. INTRODUCTION

One of the primary obstacles to directly generalizing commutative algebraic geometry to noncommutative contexts is the apparent absence of a "one-sided" Zariski topology – that is, a noncommutative Zariski topology sensitive to left (or right) module theory. In [4], the second author introduced a new "Zariski like" topology, on the set $\operatorname{Irr} R$ of isomorphism classes of simple left modules over a ring R (more generally, on the set of isomorphism classes of simple objects in a complete abelian category). This topology (see (2.4) for the precise definition), which we will refer to in this paper as the *refined Zariski topology*, satisfies the following properties (see [4]): First, when the ring R is commutative (or PI, or FBN), the topology is naturally equivalent to the classical Zariski topology on the maximal ideal spectrum of R. Second, each point (i.e., each isomorphism class of simple modules) is closed. Third, when R is noetherian, the topology is noetherian (i.e., the closed subsets satisfy the descending chain condition).

A difficulty in working with the refined Zariski topology is that it is far from obvious how to identify all the open or closed sets, although a large supply is guaranteed. In particular,

it is easily seen [**4**, 2.5(i)] that this topology is a refinement of the natural Zariski topology, in which the closed sets all have the form

$$V(I) := \ \{[N] \in \operatorname{Irr} R \mid I.N = 0\} \ ,$$

where I is an ideal of R, and where $[N]$ denotes the isomorphism class of a simple module N. Our aim is to show that, under certain relatively mild assumptions, the closed sets of the refined Zariski topology are precisely the sets

$$V(I) \ \cup \ F \ ,$$

for ideals I of R and finite subsets F of $\operatorname{Irr} R$. Under these circumstances, the refined Zariski topology can then be described as the "point closure" of the Zariski topology, that is, the coarsest topology under which the points and the Zariski closed subsets are closed.

The hypotheses under which we obtain the above results consist of a weak noetherian condition – specifically, the ACC on semiprimitive two-sided ideals – and a countability condition. The latter can be either countable vector space dimension over a field (satisfied, in particular, by any finitely generated algebra), or countable Krull dimension (in the sense of Gabriel-Rentschler). The weak noetherian hypothesis already implies that the Zariski topology on $\operatorname{Irr} R$ is noetherian, and, in the presence of either countability condition, it follows from an elementary point set topology argument that the refined Zariski topology is noetherian.

To briefly describe the contents of this paper: Section 2 presents background and general results. Section 3 concerns algebras, over a field, of countable vector space dimension, and Section 4 concerns rings with countable Krull dimension.

Throughout, R denotes an associative ring with identity.

2. Definitions and generalities

2.1 The Zariski topology. Let $\operatorname{Irr} R$ denote the set of isomorphism classes of simple left R-modules, and for each $p \in \operatorname{Irr} R$, let N_p denote a chosen simple left R-module in the equivalence class p. For ideals I of R, set

$$V(I) \ = \ \{p \in \operatorname{Irr} R \mid I.N_p = 0\} \ .$$

(Here and below, *ideal* means *two-sided ideal*.) It is not hard to see that the sets $V(I)$ constitute the closed sets of a topology on $\operatorname{Irr} R$; we will refer to this topology as the *Zariski topology* on $\operatorname{Irr} R$, and we will refer to the subsets $V(I)$ as *Zariski closed subsets* of $\operatorname{Irr} R$.

Recall that an ideal J of R is called *semiprimitive* if J is an intersection of primitive ideals, i.e., the Jacobson radical of R/J is zero. We note that each closed set of the Zariski topology has the form $V(J)$ for some semiprimitive ideal J of R, and that $V(J_1) \supsetneq V(J_2)$ for semiprimitive ideals $J_1 \subsetneq J_2$. Consequently, the Zariski topology on $\operatorname{Irr} R$ is noetherian if and only if R satisfies the ACC on semiprimitive ideals. (Semiprimitive ideals are sometimes called *J-ideals*. When using that terminology, rings with the ACC on semiprimitive ideals are called *J-noetherian* rings.)

It is also not hard to see that when R is commutative, the Zariski topology on $\operatorname{Irr} R$ is naturally equivalent to the classical Zariski topology on $\operatorname{Max} R$. On the other hand, when R is a simple ring, the Zariski topology on $\operatorname{Irr} R$ is trivial (i.e., the only closed sets are $\varnothing$ and $\operatorname{Irr} R$).

2.2 The closed-point Zariski topology. Let Y be a set equipped with a topology τ. The *point closure* of τ will refer to the coarsest topology on Y in which both the points and the τ-closed subsets are closed. We will refer to the point closure of the Zariski topology on $\operatorname{Irr} R$ as the *closed-point Zariski topology*. The following lemma pins down the point closure of a noetherian topology.

2.3 Lemma. *Let τ be a noetherian topology on a set Y, and let τ' denote the point closure of τ. Then τ' is a noetherian topology, and the τ'-closed subsets of Y are precisely the members of the collection*

$$\sigma := \{C \cup F \mid C \text{ is a } \tau\text{-closed subset of } Y, \text{ and } F \text{ is a finite subset of } Y\}.$$

Proof. We first show that σ is the collection of closed sets for a topology τ'' on Y, from which it will clearly follow that $\tau'' = \tau'$. Obviously σ contains $\varnothing$ and Y, and σ is closed under finite unions. It remains to show that σ is closed under arbitrary intersections.

Let $(X_\alpha)_{\alpha \in A}$ be a nonempty collection of sets from σ, say each $X_\alpha = C_\alpha \cup F_\alpha$ where C_α is τ-closed and F_α is finite. Since τ is noetherian, the collection of finite intersections of C_αs has a minimum element, that is, there exist indices $\alpha_1, \ldots, \alpha_n \in A$ such that every C_α contains the set $C := C_{\alpha_1} \cap \cdots \cap C_{\alpha_n}$. Now

$$C \subseteq \bigcap_{\alpha \in A} X_\alpha \subseteq X_{\alpha_1} \cap \cdots \cap X_{\alpha_n} \subseteq C \cup \left(F_{\alpha_1} \cup \cdots \cup F_{\alpha_n}\right),$$

from which we obtain $\bigcap_{\alpha \in A} X_\alpha = C \cup F$ for some $F \subseteq F_{\alpha_1} \cup \cdots \cup F_{\alpha_n}$. Since F is necessarily finite, it follows that $\bigcap_{\alpha \in A} X_\alpha \in \sigma$, as desired. Thus, σ coincides with the set of τ'-closed subsets of Y.

To see that τ' is noetherian, let $X_1 \supseteq X_2 \supseteq \cdots$ be a descending chain of τ'-closed subsets of Y. We may write each $X_i = C_i \cup F_i$ where C_i is τ-closed and F_i is finite. Note that since $X_2 \subseteq X_1$, we have $X_2 = (C_2 \cap C_1) \cup (C_2 \cap F_1) \cup F_2$, and so we may replace C_2 and F_2 by $C_2 \cap C_1$ and $(C_2 \cap F_1) \cup F_2$ respectively. Continuing inductively, we see that there is no loss of generality in assuming that $C_1 \supseteq C_2 \supseteq \cdots$. Since τ is noetherian, there is a positive integer n such that $C_i = C_n$ for $i \geq n$, and we may delete the first $n-1$ terms of our descending chain. Hence, we may assume that $C_i = C_1$ for all i.

Now each $X_i = C_1 \sqcup (F_i \setminus C_1)$, and $F_1 \setminus C_1 \supseteq F_2 \setminus C_2 \supseteq \cdots$. Since F_1 is finite, there is a positive integer m such that $F_i \setminus C_1 = F_m \setminus C_1$ for $i \geq m$, and therefore $X_i = X_m$ for $i \geq m$. $\square$

2.4 The refined Zariski topology. Let $\mathbf{A}$ be a complete abelian category, and assume that the collection $\operatorname{Irr} \mathbf{A}$ of isomorphism classes of simple objects in $\mathbf{A}$ is a set. (This assumption holds, for instance, when $\mathbf{A}$ has a generator.) For each $p \in \operatorname{Irr} \mathbf{A}$, let N_p

denote a chosen simple object in the isomorphism class p. Following [**4**], we will say that a subset $X \subseteq \operatorname{Irr} \mathbf{A}$ is *closed* provided the isomorphism class of each simple subquotient of the product

$$\prod_{p \in X} N_p$$

is contained in X. In [**4**, 2.3] it is shown that these closed sets are indeed the closed sets of a topology on $\operatorname{Irr} \mathbf{A}$, and in this paper we will call the resulting topology the *refined Zariski topology*. Note that the points are closed in this topology. In [**4**] it is proved that this topology is noetherian when $\mathbf{A}$ has a noetherian generator.

It is not hard to show that the refined Zariski topology on $\operatorname{Irr} R = \operatorname{Irr} \mathbf{Mod}\text{-}R$ is, indeed, a refinement of the Zariski topology described in (2.1). In [**4**] it is shown, further, that the refined Zariski topology on $\operatorname{Irr} R$ coincides with the Zariski topology when R is commutative, or PI, or FBN; in these cases the refined Zariski topology is again naturally equivalent to $\operatorname{Max} R$ under the classical Zariski topology.

However, it is also shown in [**4**] that the Zariski and refined Zariski topologies may differ, even in the case of the first Weyl algebra $A_1(k)$ over a field k of characteristic zero: The refined Zariski topology on $\operatorname{Irr} A_1(k)$ coincides with the finite complement topology (i.e., the nonempty open sets are the complements of finite sets), while the Zariski topology on $\operatorname{Irr} A_1(k)$ coincides with the trivial topology. In particular, in this example, the refined Zariski topology and the closed-point Zariski topology coincide.

2.5 Notational assumption. Unless otherwise noted, when we refer to $\operatorname{Irr} R$ as a topological space it is the refined Zariski topology to which we are referring.

2.6 The cofinite product condition. We isolate the key module-theoretic property used in our main results.

Let I be an ideal of R. We will say that R satisfies the *cofinite product condition at I* provided the following property holds for all families $(M_i)_{i \in \Omega}$ of pairwise nonisomorphic simple left R-modules: *If*

$$I = \operatorname{ann}_R\left(\prod_{i \in \Omega} M_i\right) = \operatorname{ann}_R\left(\prod_{i \in \Omega'} M_i\right)$$

for all cofinite subsets $\Omega' \subseteq \Omega$, then there is a left R-module embedding

$$R/I \hookrightarrow \prod_{i \in \Omega} M_i \ .$$

(Here, and below, ann_R is always used to designate the left annihilator in R.) The condition above always holds when Ω is finite, given the convention that the direct product of an empty family of modules is the zero module. Hence, the condition only needs to be checked for infinite index sets Ω.

In case the ring R/I is a left Ore domain, the cofinite product condition at I can be rephrased as follows: If $(M_i)_{i \in \Omega}$ is any family of pairwise nonisomorphic simple left

(R/I)-modules such that the product $\prod_{i \in \Omega} M_i$ is a torsion module, there exists a cofinite subset $\Omega' \subseteq \Omega$ such that the partial product $\prod_{i \in \Omega'} M_i$ is a *bounded* (R/I)-module, i.e., its annihilator in R/I is nonzero. This is a restricted version (for special families of modules) of the *left productively bounded* condition studied by Zimmermann-Huisgen and the first author in [**2**]. Several of the methods used to establish productive boundedness in [**2**] can be adapted to prove the cofinite product condition under analogous hypotheses, as we shall see in Sections 3 and 4.

We now state and prove our main abstract result, which identifies the refined Zariski topology on $\operatorname{Irr} R$ as the closed-point Zariski topology in the presence of certain restrictions on R.

2.7 Theorem. *Let R be a ring with the ascending chain condition on semiprimitive ideals, and assume that R satisfies the cofinite product condition of* (2.6) *at all of its semiprimitive prime ideals. Then each closed subset of* $\operatorname{Irr} R$ (*under the refined Zariski topology*) *is equal to $V(I) \cup S$, for some semiprimitive ideal I of R and some finite subset S of* $\operatorname{Irr} R$. *In particular, the refined Zariski and closed-point Zariski topologies on* $\operatorname{Irr} R$ *coincide, and this topology is noetherian.*

Proof. If X is a closed subset of $\operatorname{Irr} R$ and $I := \bigcap_{p \in X} \operatorname{ann}_R(N_p)$, then, since X is also a closed subset of $\operatorname{Irr}(R/I)$, we may work over the semiprimitive ring R/I. Thus, there is no loss of generality in assuming that R is semiprimitive. Second, by noetherian induction, we may (and will) assume that the conclusion of the theorem holds for all proper semiprimitive factors of R. To set up a contradiction, we assume that X is a closed subset of $\operatorname{Irr} R$ that is not equal to the union of some Zariski closed subset with some finite subset. In particular, $X \neq \operatorname{Irr} R$.

Together, the above assumptions immediately imply that (i) any closed subset (under the refined Zariski topology) contained in a proper Zariski closed subset of $\operatorname{Irr} R$ is the union of some Zariski closed subset with some finite subset; (ii) X cannot be contained in any proper Zariski closed subset of $\operatorname{Irr} R$; and (iii) any Zariski closed subset corresponding to a nonzero ideal of R must be a proper subset of $\operatorname{Irr} R$.

To start, suppose there are nonzero ideals I_1 and I_2 of R for which $I_1 I_2 = 0$. Set

$$X_1 := V(I_1) \cap X \qquad \text{and} \qquad X_2 := V(I_2) \cap X .$$

Note that $X = X_1 \cup X_2$, that X_1 and X_2 are (refined Zariski) closed subsets of $\operatorname{Irr} R$, and that $V(I_1)$ and $V(I_2)$ are proper Zariski closed subsets of $\operatorname{Irr} R$. By observation (i), X_1 and X_2 can each be written as the union of a Zariski closed subset with a finite subset. Hence, X can be written as a union of a Zariski closed subset with a finite subset, a contradiction. It follows that R is prime (and semiprimitive).

Next, set

$$M := \prod_{p \in X} N_p .$$

Note that $X \subseteq V(\operatorname{ann}_R M)$, and so $\operatorname{ann}_R M = 0$ by (ii) and (iii) above.

Now let S be an arbitrary finite subset of X. Set $Y = X \setminus S$ and

$$M' := \prod_{p \in Y} N_p \,.$$

We claim that the semiprimitive ideal $I := \mathrm{ann}_R M'$ is zero. Note that $Y \subseteq V(I)$, and so

$$X \;=\; \big(V(I) \cap X\big) \cup S \,.$$

Moreover, $V(I) \cap X$ is closed (in the refined Zariski topology), and is contained in the Zariski closed subset $V(I)$. If $I \neq 0$, then, by (iii) and (i) above,

$$V(I) \cap X \;=\; V(J) \cup T$$

for some ideal J of R and some finite subset T of $\mathrm{Irr}\, R$. But then

$$X = V(J) \cup (T \cup S) \,,$$

a contradiction. Therefore $\mathrm{ann}_R M' = 0$, as claimed.

By definition (2.4), the isomorphism class of any simple R-module subfactor of M is contained in X. However, since $\prod_{p \in Y} N_p$ is faithful for all cofinite subsets $Y \subseteq X$, the cofinite product condition (at 0) implies that R embeds as a left R-module into M. Therefore, $X = \mathrm{Irr}\, R$, another contradiction. Thus, we have proved that the closed subsets of $\mathrm{Irr}\, R$, under the refined Zariski topology, have the desired form.

Since the refined Zariski topology is a topology on $\mathrm{Irr}\, R$, it follows immediately that this topology coincides with the point closure of the Zariski topology. Moreover, since R satisfies the ACC on semiprimitive ideals, the Zariski topology on $\mathrm{Irr}\, R$ is noetherian, as noted in (2.1). Therefore, it now follows from Lemma 2.3 that the refined Zariski topology on $\mathrm{Irr}\, R$ is noetherian. □

3. Application to algebras of countable vector space dimension

In this section, we consider algebras of countable vector space dimension over a field k. We show that such algebras satisfy the cofinite product condition at all ideals, and then we apply Theorem 2.7. In fact, countable dimensional algebras satisfy a much more general cofinite product condition than that of (2.6), as follows.

3.1 Proposition. *Let R be a k-algebra with countable k-vector space dimension, I an ideal of R, and $\big(M_i\big)_{i \in \Omega}$ a family of left R-modules (not necessarily simple, and not necessarily pairwise non-isomorphic). If*

$$I \;=\; \mathrm{ann}_R\bigg(\prod_{i \in \Omega} M_i\bigg) \;=\; \mathrm{ann}_R\bigg(\prod_{i \in \Omega'} M_i\bigg)$$

for all cofinite subsets $\Omega' \subseteq \Omega$, then there is a left R-module embedding

$$R/I \hookrightarrow \prod_{i \in \Omega} M_i \, .$$

Proof. (Mimics [**2**, Proposition 3.3].) We may assume, without loss of generality, that $I = 0$. Let $b_1, b_2, \ldots$ be a k-basis for R, and set

$$M := \prod_{i \in \Omega} M_i \, ;$$

we must show that there is an embedding $_R R \hookrightarrow M$.

For $n = 1, 2, \ldots$, set $B_n := kb_1 + \cdots + kb_n$. The main step in the proof is to construct a sequence

$$\varnothing = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots$$

of finite subsets of Ω such that, for all $n = 1, 2, \ldots$, the following condition holds:

$(*)$ There exists an element $x_n \in \prod_{i \in F_n \setminus F_{n-1}} M_i$ such that $(\operatorname{ann}_R x_n) \cap B_n = 0$.

Assume that $F_0, \ldots, F_{n-1}$, for some $n > 0$, have been chosen satisfying $(*)$. Choose a nonzero element $v_1 \in B_n$. Since the module

$$\prod_{i \in \Omega \setminus F_{n-1}} M_i$$

is faithful, we can choose an index $i_1 \in \Omega \setminus F_{n-1}$ and an element $y_1 \in M_{i_1}$ such that $v_1 y_1 \neq 0$. If $(\operatorname{ann}_R y_1) \cap B_n = 0$, take $F_n = F_{n-1} \sqcup \{i_1\}$ and $x_n = y_1$. Otherwise, choose a nonzero element $v_2 \in (\operatorname{ann}_R y_1) \cap B_n \subsetneq B_n$, and then choose $i_2 \in \Omega \setminus (F_{n-1} \sqcup \{i_1\})$ and $y_2 \in M_{i_2}$ such that $v_2 y_2 \neq 0$, using the fact that

$$\prod_{i \in \Omega \setminus (F_{n-1} \sqcup \{i_1\})} M_i$$

is faithful. Continue in this manner. Since B_n is finite dimensional, we obtain, eventually, pairwise distinct indices $i_1, \ldots, i_t$ in $\Omega \setminus F_{n-1}$ and elements $y_j \in M_{i_j}$ for $j = 1, \ldots, t$ with

$$\left(\operatorname{ann}_R \{y_1, \ldots, y_t\} \right) \cap B_n \; = \; 0.$$

Set $F_n = F_{n-1} \sqcup \{i_1, \ldots, i_t\}$, and let

$$x_n \in \prod_{i \in F_n \setminus F_{n-1}} M_i$$

be the unique element such that $(x_n)_{i_j} = y_j$ for $j = 1, \ldots, t$. Then $(\operatorname{ann}_R x_n) \cap B_n = 0$, and the construction of $\varnothing = F_0 \subsetneq F_1 \subsetneq \cdots$ satisfying $(*)$ follows by induction.

To complete the proof of the proposition, let x be the unique element of M such that

$$x_i \;=\; \begin{cases} (x_n)_i & i \in F_n \setminus F_{n-1} \text{ for some } n \\ 0 & i \notin \bigcup_{n=1}^{\infty} F_n \, . \end{cases}$$

We see that $(\operatorname{ann}_R x) \cap B_n \;=\; 0$ for all $n = 1, 2, \ldots$, and so $\operatorname{ann}_R x \;=\; 0$. Therefore $_R R \cong Rx \subseteq M$. $\quad \square$

3.2 Theorem. *Let R be a k-algebra with countable k-vector space dimension, and suppose that R satisfies the ascending chain condition on semiprimitive ideals. Then each closed subset of $\mathrm{Irr}\, R$ under the refined Zariski topology is equal to $V(I) \cup S$, for some semiprimitive ideal I of R and some finite subset S of $\mathrm{Irr}\, R$. In particular, the refined Zariski and closed-point Zariski topologies on $\mathrm{Irr}\, R$ coincide, and are noetherian.*

Proof. Proposition 3.1 and Theorem 2.7. $\quad\square$

4. Application to algebras of countable Krull dimension

In this section, we consider rings with countable left (Gabriel-Rentschler) Krull dimension. We again establish the cofinite product condition and apply Theorem 2.7. As in the previous section, we establish a much more general cofinite product condition than in (2.6), although here we only establish the condition at prime ideals. Our approach largely follows [**2**] and [**3**]. The reader is referred to [**1**] and [**5**], for example, for background information on Krull dimension of modules over noncommutative rings.

4.1 Notation. Given a left R-module M, let $L(M)$ denote its lattice of submodules, and let $\kappa(M)$ denote the least ordinal κ such that the dual of the interval $[0, \kappa)$ does not embed in $L(M)$. When we write $L(R)$ or $\kappa(R)$, we regard R as a left R-module.

4.2. Let M be a left R-module. It is proved in [**3**, Theorem] that $\kappa(M)$ is countable if and only if M has countable Krull dimension (i.e., the Krull dimension of M exists and is a countable ordinal).

4.3 Lemma. *Let $\big(M_i\big)_{i \in \Omega}$ be a family of left R-modules such that*

$$\big|\{i \in \Omega \mid M_i \text{ is faithful}\}\big| \;\geq\; \big|\kappa(R)\big|.$$

Then there is an R-module embedding of $_R R$ into $M := \prod_{i \in \Omega} M_i$.

Proof. (Mimics [**2**, Proposition 5.3].) Assume that there exist no embeddings of $_R R$ into M. Consequently, $\mathrm{ann}_R\, y \neq 0$ for all $y \in M$. Set $\kappa = \kappa(R)$, and choose a subset $\Omega' \subseteq \Omega$ of cardinality κ such that M_i is faithful for all $i \in \Omega'$. Since it suffices to embed $_R R$ into $\prod_{i \in \Omega'} M_i$, there is no loss of generality in assuming that $|\Omega| = |\kappa|$ and that M_i is faithful for all $i \in \Omega$. Moreover, we may assume that $\Omega = [0, \kappa)$.

We now inductively construct elements $x_\alpha \in M_\alpha$, for $\alpha \in \Omega$, and left ideals

$$L_\gamma := \bigcap_{\alpha \leq \gamma} \mathrm{ann}_R\, x_\alpha \,,$$

for $\gamma \in \Omega$, such that $L_\beta > L_\gamma$ for all $\beta < \gamma < \kappa$. To start, choose an arbitrary $x_0 \in M_0$.

Next, let $0 < \gamma < \kappa$, and assume that x_α has been constructed for all $\alpha < \gamma$. Define $y \in M$ so that $y_\alpha = x_\alpha$ for $\alpha < \gamma$ while $y_\alpha = 0$ for $\alpha \geq \gamma$. Then,

$$\bigcap_{\alpha < \gamma} \mathrm{ann}_R\, x_\alpha \;=\; \mathrm{ann}_R\, y \;\neq\; 0 \,.$$

Now choose a nonzero element $r \in \mathrm{ann}_R \, y$, and, using the fact that M_γ is faithful, an element $x_\gamma \in M_\gamma$ such that $rx_\gamma \neq 0$. Observe that

$$L_\gamma \;=\; \bigcap_{\alpha \leq \gamma} \mathrm{ann}_R \, x_\alpha \;<\; \bigcap_{\alpha < \gamma} \mathrm{ann}_R \, x_\alpha \;\leq\; \bigcap_{\alpha \leq \beta} \mathrm{ann}_R \, x_\alpha \;=\; L_\beta$$

for all $\beta < \gamma$. This establishes the induction step, and the construction is complete.

To conclude the proof, observe that the rule $\gamma \mapsto L_\gamma$ defines a map $[0, \kappa) \to L(R)$ such that $L_\beta > L_\gamma$ for all $\beta < \gamma < \kappa$. However, the existence of such a map contradicts the definition of κ. The lemma is proved. $\square$

4.4 Lemma. *Let M be a left R-module and $\big(M_i\big)_{i \in \Omega}$ a family of nonzero submodules of M. Set*

$$S = \left\{ \Omega' \subseteq \Omega \;\Big|\; \bigcap_{j \in \Omega'} M_j \neq 0 \right\},$$

and assume that

    (a) $\Omega \notin S$,
    (b) S *is closed under finite unions,*
    (c) *if Ω' and Ω'' are any disjoint subsets of Ω, then either $\Omega' \in S$ or $\Omega'' \in S$.*

Then $\kappa(M)$ is uncountable.

Proof. This is a restatement of [**2**, Lemma 5.4]. $\square$

4.5 Proposition. *Let R be a ring with countable left Krull dimension, I a prime ideal of R, and $\big(M_i\big)_{i \in \Omega}$ a family of left R-modules (not necessarily simple, and not necessarily pairwise non-isomorphic). If*

$$I \;=\; \mathrm{ann}_R\left(\prod_{i \in \Omega} M_i \right) \;=\; \mathrm{ann}_R\left(\prod_{i \in \Omega'} M_i \right)$$

for all cofinite subsets $\Omega' \subseteq \Omega$, then there is a left R-module embedding

$$R/I \;\hookrightarrow\; \prod_{i \in \Omega} M_i.$$

Proof. (Mimics [**2**, Theorem 5.5].) Without loss of generality, we may assume that R is prime and $I = 0$. We must show that $_R R$ embeds into the module

$$M := \prod_{i \in \Omega} M_i \;.$$

To start, by (4.2), $\kappa(R)$ is countable. Consequently, if infinitely many M_i are faithful, then the result follows from Lemma 4.3. Hence, we may assume there exist at most finitely many i such that M_i is faithful, and we may remove these indices from Ω without loss of

generality. In particular, we can assume that $A_i := \operatorname{ann}_R M_i \neq 0$ for all $i \in \Omega$. Because M is faithful,

$$\bigcap_{i \in \Omega} A_i = 0 \, .$$

Now set

$$S \; = \; \left\{ \Omega' \subseteq \Omega \; \middle| \; \bigcap_{i \in \Omega'} A_i \neq 0 \right\} \, .$$

Since R is prime, S is closed under finite unions. Because $\kappa(R)$ is countable, it then follows from Lemma 4.4 that there exist disjoint subsets $\Gamma_1, \Lambda_1 \subseteq \Omega$ which do not belong to S, that is,

$$\bigcap_{i \in \Gamma_1} A_i \; = \; \bigcap_{i \in \Lambda_1} A_i \; = \; 0 \, .$$

Similarly, the collection $\{\Omega' \in S \mid \Omega' \subseteq \Lambda_1\}$ is also closed under finite unions, and so Lemma 4.4 implies that there are disjoint subsets $\Gamma_2, \Lambda_2 \subseteq \Lambda_1$ such that

$$\bigcap_{i \in \Gamma_2} A_i \; = \; \bigcap_{i \in \Lambda_2} A_i \; = \; 0 \, .$$

Continuing inductively, we obtain subsets $\Gamma_1, \Lambda_1, \Gamma_2, \Lambda_2, \cdots \subseteq \Omega$ such that

(a) $\Gamma_n \cap \Lambda_n = \varnothing$
(b) $\Gamma_{n+1} \cup \Lambda_{n+1} \subseteq \Lambda_n$
(c) $\bigcap_{i \in \Gamma_n} A_i = \bigcap_{i \in \Lambda_n} A_i = 0$

for all n. Note that the sets $\Gamma_1, \Gamma_2, \ldots$ are pairwise disjoint.

By construction, each of the modules

$$P_n := \prod_{i \in \Gamma_n} M_i$$

is faithful. Since $|\mathbb{N}| \geq |\kappa(R)|$, it follows from Lemma 4.3 that there is a left R-module embedding

$$R \; \hookrightarrow \; \prod_{n \in \mathbb{N}} P_n \; =: P \, .$$

This completes the proof, because P embeds as a left R-module into M. $\quad \square$

4.6 Theorem. *Let R be a ring with countable left Krull dimension, and suppose that R satisfies the ascending chain condition on semiprimitive ideals. Then each closed subset of $\operatorname{Irr} R$ under the refined Zariski topology is equal to $V(I) \cup S$, for some semiprimitive ideal I of R and some finite subset S of $\operatorname{Irr} R$. In particular, the refined Zariski and closed-point Zariski topologies on $\operatorname{Irr} R$ coincide, and are noetherian.*

Proof. Proposition 4.5 and Theorem 2.7. $\quad \square$

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