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Research Statement

1 Overview

My research deals with the stability of incompressible, inviscid fluid flows. In essence, the question of stability for fluids is this: If we perturb our initial conditions by a small amount, will the new solution remain close to the original one? This question is important because, if a mathematically predicted flow is not stable we will not see it physically. The finite dimensional analog is something like a marble sitting on the top of an inverted bowl. This marble is at an equilibrium, so the equations of motion for the marble predict that it will remain rested at the top of the bowl. However, this equilibrium is unstable, so the slightest perturbation will cause the marble to roll off the bowl. An equilibrium fluid flow has a velocity field that does not change with time. There are many approaches to determining the stability of a fluid flow, the approach I take is to examine the so-called linear stability of equilibrium flows. To do this, we must linearize the equations of motion describing the evolution of a perturbation to our equilibrium. In my work I split the space of possible perturbations into two classes and develop criteria for linear instability for each of these classes.

In addition to studying the stability of Euler's equation, I have also implemented existing code to compute numerical solutions to 2D incompressible flows, both inviscid (Euler's equation) and viscous (Navier-Stokes). I plan to study the relationship between the analytic and numerical solutions.

My future research plans also include supervising undergraduate research, so I discuss some ideas for projects in the last section of this document.

2 Thesis Research

2.1 Linear instability criteria

The flow of a fluid is described by a vector field v, where the vector $v(x,t)$ is the velocity of a fluid particle traveling through point x at time t. I study incompressible fluids; this assumption imposes the mathematical requirement that the divergence of the velocity field is 0, $\partial_k v^k = 0$, at all times. The divergence-free requirement leads to the introduction of a scalar pressure $p = p(x,t)$ into the description of our fluid flow. Whereas, the Navier-Stokes equation describes viscous flows, the flows I consider have no viscosity. Thus, the evolution of the velocity v and pressure p is governed by Euler's equation:

$$\begin{cases} \partial_t v^j = -v^k \cdot \partial_k v^j - \partial_j p \\ \partial_k v^k = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{1}$$

Here v_0 is the initial (divergence-free) velocity field. My fluid domain is $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ for $n = 2,3$; this corresponds to periodic boundary conditions.

We introduce the notion of linear stability only for 'steady-state' solutions, i.e. solutions of (1) that do not depend on time. These are analogs of critical points of a dynamical system.

Appearently, a steady-state solution u, p satisfies the following system:

$$\begin{cases} u^k \cdot \partial_k u^j = -\partial_j p \\ \partial_k u^k = 0 \end{cases} \tag{2}$$

If we impose periodic boundary conditions, there are many solutions to (2). To define reasonable criteria for linear instability of a given steady-state solution $u \in C^\infty(\mathbb{T}^n)$, we linearize Euler's equation (1) at u and get the following equation for the linear evolution of a divergence-free perturbation w with scalar pressure q:

$$\begin{cases} \partial_t w = -u^k \cdot \partial_k w^j - w^k \cdot \partial_k u^j - \partial_j q \\ w(x,0) = w_0(x) \end{cases} \tag{3}$$

This system has a unique solution for each divergence-free initial perturbation $w_0 \in L^2(\mathbb{T}^n)$; hence, we may define a linear evolution operator that maps initial conditions of (3) to solutions at a given time. Let $L^2_{sol}(\mathbb{T}^n)$ denote the space of divergence-free vector fields in $L^2(\mathbb{T}^n)$ and define the evolution operator $G(t) : L^2_{sol}(\mathbb{T}^n) \to L^2_{sol}(\mathbb{T}^n)$, by $G(t)w_0(x) = w(x,t)$.

The concept of linear stability, which is implicitely defined below (Definition 2), requires that the perturbation w does not grow too much as time increases. To measure the growth of our perturbations, we will use the notion of spectrum of a linear operator.

Definition 1. *Let X be a Banach space and $T : X \to X$ be a bounded linear operator. Then the spectrum of T, denoted $\sigma(T)$, is the set of points $\lambda \in \mathbb{C}$ such that the operator $\lambda I - T$ does not have a bounded inverse.*

For example the spectrum of a matrix is its set of eigenvalues. Just as in the finite dimensional case, the spectrum of an infinite dimensional operator gives us some indication of how the operator stretches vectors in its domain. To see how the spectrum affects the growth of a general bounded linear operator T, we consider the radius of the spectrum: $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Definition 2. *Let u be a solution to time-independent Euler's equation (2), and let $G(t)$ be its associated evolution operator. We say u is linearly unstable if for some time $t_0 > 0$, $r(G(t_0)) > 1$.*

In my work I look at a particular subset of the spectrum for $G(t)$ called the essential spectrum. The essential spectrum consists of all points of the spectrum that do not behave like eigenvalues with finite multiplicity. One can demonstrate the instability of some equilibrium u if one finds that the radius of the essential spectrum of $G(t)$, denoted $r_{ess}(G(t))$, is greater than 1 for some $t > 0$.

My work is an extention of work done by Vishik in [V], establishing a method for computing $r_{ess}(G(t))$. Growth in the essential spectrum can be connected to the growth of a vector-valued function of time, $b(t)$, which corresponds to solutions of a related PDE along characteristics in the cotangent space $T^*(\mathbb{T}^n)$. The vector-valued function b is defined to be a solution to the following system of ODEs (called the bicharacteristic amplitude system):

$$\begin{cases} \dot{x} = u(x) \\ \dot{\xi} = -\left(\frac{\partial u}{\partial x}\right)^T \xi \\ \dot{b} = -\left(\frac{\partial u}{\partial x}\right) b + 2\left(\frac{\partial u}{\partial x} b, \xi\right) \frac{\xi}{|\xi|^2} \end{cases} \tag{4}$$

If we let $b(x_0, \xi_0, b_0; t)$ be the solution to (4) with initial conditions $x(0) = x_0$, $\xi(0) = \xi_0$ and $b(0) = b_0$, define the (Lyapunov) exponent μ to measure the maximum growth of all solutions to (4):

$$\mu := \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0, |\xi_0|=|b_0|=1 \\ b_0 \perp \xi_0}} |b(x_0, \xi_0, b_0; t)|, \tag{5}$$

Theorem 2.1 (M.M. Vishik)**.** *Let u be a time-independent solution to Euler's equation and $G(t)$ its associated evolution operator. Then for μ defined in equation (5), $r_{ess}(G(t)) = e^{\mu t}$.*

2.2 Two classes of perturbations

My thesis research involves separating perturbations into two classes and exploring the growth rate for each class. In his book, *Mathematical Methods of Classical Mechanics*, V.I. Arnold characterizes the dynamics of an incompressible, inviscid fluid in a geometric way. We may think of the motion of a fluid as a family of volume preserving diffeomorphisms of the fluid domain indexed by time. These diffeomorphisms form an infinite-dimensional Lie group. If we let kinetic energy be the metric for our Lie group, then geodesics will correspond to flows that minimize kinetic energy. Thus we may view fluid dynamics as motion along these geodesics. The two classes of perturbations that are studied in my thesis research are most naturally described in this geometric view of fluid dynamics. The first space of perturbations I study is the orbit of our steady solution, u, under the co-adjoint action of the group; we denote this space V_u. These orbits are of some fundamental interest in the study of fluids; one of the essential differences between 2D and 3D fluid dynamics is the difference in the geometries of these orbits in the two cases, see [A]. The second space of perturbations I consider are those in the canonical factor space L^2_{sol}/V_u.

2.3 Lower bounds for the essential spectral radius for both classes

I have established a lower bound for the essential spectral radius of the evolution operator on V_u for 3D flows (Theorem 2.2) and, under a certain condition on the support of curl$u$, a lower bound for the essential spectral radius of the evolution operator on the factor space for 2D and 3D flows (Theorem 2.4). Both results are similar to Theorem 2.1 in spirit: each lower bound is in terms of a Lyapunov-type exponent related to the same bicharacteristic amplitude system that appears in [V].

Theorem 2.2 (Thoren)**.** *Let $G(t)$ be the solution operator for 3D Euler's equation with periodic boundary conditions linearized at a steady solution $u \in C^\infty(\mathbb{T}^3)$, and let*

$$\mu_I := \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0, |\xi_0|=|b_0|=1 \\ b_0 \perp \xi_0 \\ x_0 \in supp(curl u)}} |b(x_0, \xi_0, b_0; t)|.$$

where $b(x_0, \xi_0, b_0; t)$ is a solution to equation (4) of BAS corresponding to initial conditions $x(0) = x_0$, $\xi(0) = \xi_0$ and $b(0) = b_0$. Then $e^{\mu_I t} \le r_{ess}(G(t)|_{\overline{ImB}})$.

The exponent μ_I differs from the original exponent μ in that the extra assumption $x_0 \in$ supp(curl$u$) is placed on the supremum. This leads to the following corollary.

Corollary 2.3. *For 3D flows on the torus, if the support of curlu is the entire fluid domain, $\mathbb{T}^3$, then*

$$r_{ess}(G(t)\mid_{V_u}) = r_{ess}(G(t)).$$

I have also computed a lower bound for growth in the factor space for the case of 2D and 3D flows whenever supp(curl$u$) is a proper subset of the fluid domain, $\mathbb{T}^n$.

Theorem 2.4 (Thoren)**.** *Let $G(t)$ be the solution operator for Euler's equation with periodic boundary conditions linearized at a steady solution $u \in C^\infty(\mathbb{T}^n)$. Also assume supp(curlu) $\subsetneq \mathbb{T}^n$. If we let $G_F(t)$ denote this operator acting on the factor space $L^2_{sol}/\overline{ImB}$ and define*

$$\mu_F := \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0, |\xi_0|=|b_0|=1 \\ b_0 \perp \xi_0 \\ x_0 \notin supp(curlu)}} |b(x_0, \xi_0, b_0; t)|.$$

where $b(x_0, \xi_0, b_0; t)$ is a solution to equation (4) of BAS corresponding to initial conditions $x(0) = x_0$, $\xi(0) = \xi_0$ and $b(0) = b_0$. Then $e^{\mu_F t} \leq r_{ess}(G_F(t))$.

The Lyapunov-type exponent here μ_F differs from μ in that the condition $x_0 \notin$ supp(curl$u$) has been added to the supremum. The connection between the new Lyapunov exponents, μ_I and μ_F, and the original Lyapunov exponent μ leads to this next corollary.

Corollary 2.5. *For flows in 3D if supp(curlu) is a proper subset of $\mathbb{T}^3$, then*

$$r_{ess}(G(t)) = \max\{r_{ess}(G_F(t)), r_{ess}(G(t)\mid_{\overline{Im\,B}})\}.$$

Thus, I have shown that any instability associated with the essential spectrum is realized in one of the two classes of perturbations.

Currently I am working on expanding these results. In particular, I am trying to find a way to compute an upper bound for $r_{ess}(G(t)\mid_{V_u})$ for general 3D flows and find an example of a 3D equilibrium flow where $r_{ess}(G(t)\mid_{V_u}) < r_{ess}(G(t))$. I would also like to discover more about growth in both classes of perturbations for 2D flows. Methods similar to these may be useful in analyzing the essential spectrum of the linear evolution operator associated with vorticity; this is one possible direction for future research. My longer term goal is to use new knowledge of the spectrum of the linear evolution operator to better understand when the linear instablility of a steady-state solution implies nonlinear instability of that solution.

3 Computational Project

This past spring I worked with Yingda Cheng, under the supervision of Irene Gamba, computing numerical solutions to 2D Euler equation and Navier-Stokes equation. We used a discontinuous Galerkin method for computing the vorticity of a 2D incompressible fluid flow developed by Liu and Shu in [LS]. I have successfully implemented the Fortran code written by Liu to compute numerical solutions for Euler's equation with periodic boundary conditions and am now able to incorporate new code for different initial conditions. This project will continue next spring and we hope to analyze some specific numerical solutions in detail.

4 Undergraduate Research

To my mind, there are two motivations for undergraduate research: (1) to expose students to valuable or interesting subject matter not covered in traditional mathematics courses in a way that facilitates greater appreciation for and comfort with mathematical concepts, and (2) to give especially talented students the opportunity to work to the limits of their abilities and perhaps create original research. The ultimate goal in both cases is to foster a love for mathematics and build critical thinking skills at a level beyond that of the classroom. I look forward to supervising research projects of both flavors.

If the goal of the research project is more in keeping with motivation (1), the choice of project would likely depend on the particular student's taste. If a student is more inclined towards algebraic concepts, I might suggest looking at quaternions and their connection to 3D rotations or an extension of topics covered in linear algebra. If the student has interests in physics or appreciates more 'visualizable' questions I would suggest some topics connecting calculus with physics–perhaps an in-depth look at the variations of Stokes' Theorem. If the student feels more comfortable with experimentation or programming I would suggest a project utilizing computer programs (or developing new programs) that explore ideas the analytic tools available to the student cannot reach.

I also have project ideas more closely related to my own research to challenge the especially talented student. Some aspects of stability for plane parallel shear flow (i.e. flow straight down a pipe) only require vector calculus and basic ODEs. I would also love to introduce the functional calculus of bounded self-adjoint operators to an undergraduate student. The subject is so algebraic in nature that many of the key ideas do not require heavy background in analysis.

I am eager to engage students of *all* preparation levels in mathematical projects that not only build their critical thinking skills, but also their appreciation for a beautiful subject. In some cases this is about challenging students in way that truly engages their interest. In other cases (especially for women and underrepresented minorities) this is a matter of increasing a student's confidence in their own abilities.

References

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[LS] J.-G. Liu and C.-W. Shu, A high-order discontinuous Galerkin method for 2D incompressible flows, *J. Comput. Phys.* **160**, (2000), 577-596.

[V] M.M. Vishik, Spectrum of small oscillations of an ideal fluid and Lyapunov exponents, *J. Math. Pures Appl.* **75** (1996), 531-557.