Some remarks on the moduli of K3 surfaces

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A K3 surface is a compact complex surface X which is connected and simply connected and has trivial canonical bundle K_X. A K3 surface X is Kählerian if there exist Kähler metrics on X; we will call X Kähler only if the metric has been specified. The moduli of K3 surfaces has been extensively studied by using the period map: if a basis $\gamma_1, \dots, \gamma_{22}$ has been chosen for $H_2(X,\mathbb{Z})$, the periods of a holomorphic 2-form ω_X on X give a well-defined point

$$[\int_{\gamma_1}\omega_X, \dots, \int_{\gamma_{22}}\omega_X] \in \mathbb{P}^{21}.$$

This approach goes back to work of Andreotti and Weil (cf. [36]). Using a refinement of this period map, Burns and Rapoport [6] have constructed a moduli space for (marked) Kählerian K3 surfaces; their work is reviewed in section 1.

This paper has three goals. First, we will discuss a "moduli space" for (marked) Kähler (rather than Kählerian) K3 surfaces. Strictly speaking, the corresponding moduli functor is not representable, but we will construct a real analytic manifold M and a class of maps from complex analytic varieties to M which in some sense represents the moduli functor. In contrast with the Burns-Rapoport space, M is separated, and the integral automorphism group Γ of the K3 lattice operates properly discontinuously on M, so that we may form a "coarse moduli space" M/Γ of (unmarked) Kähler K3 surfaces. (A similar construction provides in general a good period space for Hodge structures

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of weight 2, with $h^{2,0} = 1$.)

Second, we will construct extensions of the period maps to families of K3 surfaces which acquire rational double points. This is essential when studying families of algebraic K3 surfaces, and will allow us to regard the moduli spaces of marked algebraic K3 surfaces as "subspaces" of a space $\overline{M}$ (which contains the moduli space M above as an open subset). The embeddings of these subspaces are compatible with the actions of the integral automorphism groups.

Our third goal is to prove a sharpened version of Kulikov's theorem [14] on the surjectivity of the period maps for algebraic K3 surfaces. This version was used by Todorov [33] in his proof of the surjectivity of the Burns-Rapoport period map for Kählerian K3 surfaces.

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1. Period maps for K3 surfaces

Fix a free $\mathbb{Z}$-module L of rank 22 which has an even unimodular symmetric bilinear form of signature (3, 19). (All such lattices L are isometric: cf. [30]). We let $L_{\mathbb{R}} = L \otimes \mathbb{R}$ and $L_{\mathbb{C}} = L \otimes \mathbb{C}$. If X is a K3 surface, there exist isometries $\alpha: H^2(X,\mathbb{Z}) \xrightarrow{\sim} L$; a choice of such an isometry is called a marking of X. The classical notion of periods of a marked K3 surface arises in the following way: the isometry α determines the subspace $H^{2,0}(X) \subset H^2(X,\mathbb{C}) \xrightarrow{\sim} L_{\mathbb{C}}$. $H^{2,0}(X)$ is a $\mathbb{C}$-vector space of dimension one, and if $\omega_X \in H^{2,0}(X)$ is a generator, then $\langle\omega_X,\omega_X\rangle = 0$ and $\langle\omega_X,\overline{\omega}_X\rangle > 0$. We can thus associate to (X,α) a point in the classical period domain

$$\Omega = \{ \omega \in L_{\mathbb{C}} \mid \langle\omega,\omega\rangle = 0, \langle\omega,\overline{\omega}\rangle > 0 \} / \mathbb{C}^* \subset \mathbb{P}^{21}.$$

(Note that Ω is a complex manifold of dimension 20.) Every point $x \in \Omega$ determines a Hodge structure of weight 2 on $L_{\mathbb{C}}$ as follows: if $\omega \in L_{\mathbb{C}}$ is a representative of x, define

$$H^{2,0}(x) = \mathbb{C}\omega \subset L_{\mathbb{C}}$$
$$H^{0,2}(x) = \mathbb{C}\overline{\omega} \subset L_{\mathbb{C}}$$
$$H^{1,1}(x) = (H^{2,0}(x) \oplus H^{0,2}(x))^{\perp} \subset L_{\mathbb{C}}.$$

Let $p\colon \mathcal{X} \to S$ be a family of K3 surfaces. A marking in this case is an isomorphism of local systems

$$\alpha\colon R^2p_*(\mathbb{Z}) \tilde{\to} \underline{L}_S.$$

A marked family of K3 surfaces thus has a classical period map $\tau_S\colon S \to \Omega$ which associates to each marked fiber (X_s,α_s) the corresponding point of Ω. This is a holomorphic map, and if $(\mathcal{X},X) \to (S,*)$ is a local universal deformation of the K3 surface X, the corresponding classical period map $S \to \Omega$ is a local isomorphism at $*$ (the "local Torelli theorem").

Unfortunately, certain information about families $\mathcal{X} \to S$ is "lost" by the classical period map: it is possible to have two non-isomorphic families with the same classical period map. For Kählerian K3 surfaces, this defect is remedied by the "Burns-Rapoport period map."

Define two real manifolds $M \subset \overline{M}$ by

$$\overline{M} = \{ (\omega,\kappa) \in \Omega \times L_{\mathbb{R}} \mid \langle\omega,\kappa\rangle = 0 \text{ and } \langle\kappa,\kappa\rangle = 1 \},$$

and

$$M = \{ (\omega,\kappa) \in \overline{M} \mid \text{for all } \delta \in L \text{ with } \langle\delta,\delta\rangle = -2, \text{ if } \langle\omega,\delta\rangle = 0 \text{ then } \langle\kappa,\delta\rangle \neq 0 \}.$$

(We call M the <u>polarized period domain</u> and $\overline{M}$ the <u>weakly polarized period domain.</u>) Define an equivalence relation on M by setting $(\omega,\kappa) \sim (\omega,\kappa')$ if κ and κ' are in the same connected component of the fiber $pr_1^{-1}(\omega) \subset M$. The <u>Burns-Rapoport period domain</u> is the space

$$\tilde{\Omega} = M / \sim.$$

Burns and Rapoport [6] prove that $\tilde{\Omega}$ is a (non-separated) complex analytic space, and the induced map $\pi\colon \tilde{\Omega} \to \Omega$ is étale. A point $x \in \tilde{\Omega}$ corresponds to

1) the Hodge structure determined by $\pi(x)$
2) a choice V^+ of one of the connected components of

$$V = \{ \kappa \in H^{1,1} \cap L_{\mathbb{R}} \mid \langle\kappa,\kappa\rangle = 1 \}$$

3) a partition $P = \Delta^+ \cup -\Delta^+$ of the set

$$\Delta = \{ \delta \in H^{1,1} \cap L \mid \langle\delta,\delta\rangle = -2 \}$$

such that

a) if $\delta_1, \ldots, \delta_k \in \Delta^+$ and $\delta = \Sigma\, n_i\delta_i \in \Delta$ with $n_i \geq 0$ then $\delta \in \Delta^+$, and

b) $V_P^+ = \{ \kappa \in V^+ \mid \langle\kappa,\delta\rangle > 0 \text{ for all } \delta \in \Delta^+ \} \neq \emptyset$.

These data satisfy a "continuity condition:" for every $x \in \tilde{\Omega}$ and every $\kappa \in V_P^+(x)$, there is an open neighborhood K of κ in $L_{\mathbb{R}}$ and an open neighborhood U of x in $\tilde{\Omega}$ such that for every $y \in U$,

$$\Delta^+(y) = \{ \delta \in \Delta(y) \mid \langle\kappa,\delta\rangle > 0 \text{ for all } \kappa \in K \}.$$

The <u>Burns-Rapoport period map</u> associates to a marked Kählerian K3 surface (X,α) the point of $\widetilde{\Omega}$ determined by

1) the Hodge structure of X
2) the component $V^+(X)$ of V containing the cohomology class of any Kähler metric on X
3) $\Delta^+(X) = \{ \delta \in \Delta(X) \mid \delta \text{ is an effective divisor on } X \}$.

Notice that with this definition, we have

$$\Delta(X) = \{ \delta \in H^{1,1}(X) \cap L \mid \langle\delta,\delta\rangle = -2 \}, \text{ and}$$

$$V_P^+(X) = \{ \kappa \in V^+(X) \mid \langle\kappa,\delta\rangle > 0 \text{ for all effective } \delta \in \Delta(X) \}$$

By Riemann-Roch, $\pm\,\delta$ is effective for each δ $\Delta(X)$ so that

$$\Delta^+(X) \cup -\Delta^+(X)$$

is a partition of $\Delta(X)$ as required.

If $\mathfrak{X} \to S$ is a marked family of Kählerian K3 surfaces, Burns and Rapoport prove that the induced map $\widetilde{\tau}_S \colon S \to \widetilde{\Omega}$ is a complex analytic map. For this period map, we have the

<u>Burns-Rapoport Global Torelli Theorem</u> ([6] and [18]).

Two smooth marked Kählerian K3 surfaces X and X' with the same Burns-Rapoport periods are isomorphic. More precisely, if $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is an isometry which preserves the Hodge structures, maps $V^+(X)$ to $V^+(X')$ and $\Delta^+(X)$ to $\Delta^+(X')$, then there is a unique isomorphism $\Phi\colon X \xrightarrow{\sim} X'$ with $\Phi^* = \phi$.

Let us indicate briefly how the Burns-Rapoport global Torelli theorem is used in the study of the fine moduli space for marked Kählerian K3 surfaces (cf. [6] or [18] for more details). Consider the functor which associates to each complex analytic space S the set of isomorphism classes of pairs

$$(p: \mathfrak{X} \to S, \quad \alpha: R^2p_*(\mathbb{Z}) \tilde{\to} \underline{L}_S)$$

where p: $\mathfrak{X} \to S$ is a family of Kählerian K3 surfaces, and α is a marking of the family. For each Kählerian K3 surface X, there is a local universal deformation p: $(\mathfrak{X}, X) \to (S,*)$ which has a natural marking α: $R^2p_*(\mathbb{Z}) \tilde{\to} H^2(X,\mathbb{Z})$. If we choose an isomorphism $H^2(X,\mathbb{Z}) \cong L$, we get a period map $\tilde{\tau}_S$: $S \to \tilde{\Omega}$, and the Torelli theorem guarantees that $\tilde{\tau}_S$ is injective. A fine moduli space is now constructed by glueing the versal families $\mathfrak{X} \to S$ together by using the period maps $\tilde{\tau}_S$; this space can be identified with an open subset $\tilde{\Omega}_0 \subset \tilde{\Omega}$, which has a universal marked family $\mathfrak{X}_{\tilde{\Omega}_0} \to \tilde{\Omega}_0$.

2. The moduli of Kähler K3 surfaces

There are two disadvantages inherent in the Burns-Rapoport period map. The first is its compatibility with period maps for algebraic K3 surfaces: as will be discussed in section 5, there is no (canonical) way to embed the moduli spaces of marked algebraic K3 surfaces in the Burns-Rapoport space. The second problem is that for the Burns-Rapoport period map (as well as the classical period map), the marking is absolutely essential: the automorphism group Γ of the lattice L does not operate in a properly discontinuous fashion on $\tilde{\Omega}$ or Ω. We thus introduce a polarized period map for Kähler K3 surfaces.

Let X be a Kählerian K3 surface. A polarization[2] on X is a

[2] Our use of the term polarization is a departure from previous usage: our polarizations are not required to be integral.

class $\kappa \in V_p^+(X)$, in other words, a real (1,1) class which has positive intersection with every effective divisor and lies in $V^+(X)$. If X is a Kähler K3 surface, then the cohomology class of the Kähler metric (after suitable renormalization) gives a polarization κ; such a polarization is called a <u>Kähler polarization</u>. We shall assume henceforth that all of our Kähler metrics are normalized so that $\langle\kappa,\kappa\rangle = 1$.

If p: $\mathfrak{X} \to S$ is a family of Kählerian K3 surfaces with a marking α: $R^2p_*(\mathbb{Z}) \xrightarrow{\sim} \underline{L}_S$, then a <u>polarization</u> on the family is a section $\kappa \in \Gamma(S, \underline{L}_S \otimes \mathbb{R}) \cong L_{\mathbb{R}}$ such that $\kappa|_s$ is a polarization on X_s for every $s \in S$. κ is <u>Kähler</u> if $\kappa|_s$ is Kähler for every $s \in S$. One common way to obtain a Kähler polarization is from a Kähler metric on $\mathfrak{X}$: if $\mathfrak{X}$ is a Kähler manifold and p is smooth, the family of induced Kähler metrics gives a polarization on $\mathfrak{X} \to S$.

Recall that in section 1 we defined the polarized period domain M and the weakly polarized period domain $\overline{M}$ with $M \subset \overline{M} \subset \Omega \times L_{\mathbb{R}}$. The fibers of the map $\overline{M} \to \Omega$ are real analytic manifolds of dimension 19, so that $\overline{M}$ is not a complex manifold. However, for a complex analytic space S, we say that f: $S \to \overline{M}$ is <u>quasi-analytic</u> if

1) the composite $(pr_1 \circ f)$: $S \to \Omega$ is analytic, and
2) the composite $(pr_2 \circ f)$: $S \to L_{\mathbb{R}}$ is constant.

Suppose that $\mathfrak{X} \to S$ is a family of K3 surfaces with a marking α: $R^2p_*(\mathbb{Z}) \xrightarrow{\sim} \underline{L}_S$ and a polarization $\kappa \in \Gamma(S, \underline{L}_S \otimes \mathbb{R}) \cong L_{\mathbb{R}}$. The classical period map together with κ gives a map $S \to \Omega \times L_{\mathbb{R}}$, whose image is contained in M; we call the induced map τ_S^p: $S \to M$ the <u>polarized period map</u>. Regarded as a map $S \to \overline{M}$, this is clearly a quasi-analytic map.

For this period map, we have the

Polarized Global Torelli Theorem

Let (X,κ) and (X',κ') be two K3 surfaces with Kähler polarizations. Suppose that $\phi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is an isometry preserving the Hodge structure such that $\phi(\kappa) = \kappa'$. Then there is a unique isomorphism $\Phi: X \tilde{\to} X'$ with $\Phi^* = \phi$.

Proof: This follows immediately from the Burns-Rapoport Global Torelli theorem, and the definitions. Q.E.D.

When κ and κ' are algebraic polarizations (that is, when the Kähler metrics are (renormalized) Hodge metrics), then the global Torelli theorem in this form was first proved by Piateskii-Shapiro and Shafarevich [25].

Consider now the moduli functor which associates to each complex analytic space S the set of isomorphism classes of triples

$$(p: \mathfrak{X} \to S, \quad \alpha: R^2p_*(\mathbb{Z}) \tilde{\to} \underline{L}_S, \quad \kappa \in \Gamma(S, \underline{L}_S \otimes \mathbb{R}))$$

of marked families of K3 surfaces with a Kähler polarization. This functor is not representable in the category of complex analytic spaces. However, there is an open subset $M_0 \subset M$, and a family $\mathfrak{X}_{M_0} \to M_0$ with the property that the set of quasi-analytic maps $S \to M_0$ (together with the induced families of marked polarized K3 surfaces) coincides with our functor. $\mathfrak{X}_{M_0} \to M_0$ is constructed in the same way as the moduli space for Kählerian K3 surfaces: for each Kähler K3 surface X (with Kähler polarization κ), there is a local universal deformation $(\mathfrak{X},X) \to (S,*)$. Moreover, there is a hypersurface $T \subset S$ such that on each fiber of the restricted family $\mathfrak{X}_T \to T$, κ is the cohomology class of a Kähler metric. We use the period maps $\tau_T^P: T \to M$ to glue together the families $\mathfrak{X}_T \to T$ as before, and obtain the "moduli space" $M_0 \subset M$.

Let X_{diff} denote the underlying C^∞ manifold of a K3 surface. The moduli space M_0 is closely related to the moduli space N_0 of Ricci-flat Kähler metrics on X_{diff} which has been studied by Bourguignon [3] and Todorov [34]. By a theorem of Yau [37], [38], every Kähler metric on a K3 surface is cohomologous to a unique Ricci-flat Kähler metric (so that there is a Ricci-flat Kähler metric corresponding to each Kähler polarization). If we fix the Ricci-flat metric, the set of complex structures for which it is a Kähler metric forms a complex line $\mathbb{CP}^1 \subset M_0$. $M_0 \to N_0$ is in fact a C^∞ fiber bundle with fiber $\mathbb{CP}^1$, and sits inside a similar fiber bundle $M \to N$.

3. Families of surfaces with rational double points

Let (X,P) be a germ of a surface with a rational double point, and let $\rho: (Y,C) \to (X,P)$ be the minimal resolution of singularities. We write $\rho^{-1}(P) = C = \Sigma\, C_i$ as a sum of irreducible components, and let $\Lambda = \Lambda(X)$ be the free abelian group generated by $\{C_i\}$. Λ has a quadratic form $<\ ,\ >$ induced by the intersection form on C, and we may identify Λ with $H^2(Y,\mathbb{Z})$. We let $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ and $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$. If we define

$$R = R(X) = \{\ \delta \in \Lambda \mid <\delta,\delta> = -2\ \},$$

then $R \subset \Lambda_{\mathbb{R}}$ is a root system (following the definitions[3] of Bourbaki [2]) and $C_i \in R$ for each i. For each $\delta \in R$, let $s_\delta \in \mathrm{Aut}(\Lambda)$ be the reflection in δ:

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[3] Actually, our definition differs from Bourbaki's in that for us, the bilinear form on a root system is negative (rather than positive) definite; we have simply changed the sign on the bilinear form.

$$s_\delta : \lambda \to \lambda + \langle \lambda, \delta \rangle \, \delta$$

and let W be the subgroup of $\mathrm{Aut}(\Lambda)$ generated by $\{ s_\delta \mid \delta \in R \}$; W is the <u>Weyl group</u> of the root system.

Let $(\mathcal{Y}, Y) \to (\mathrm{Def}_Y, *)$ and $(\mathcal{X}, X) \to (\mathrm{Def}_X, *)$ be the versal deformations. In any deformation of Y, a contraction can be made to yield a deformation of X [29]; this gives a natural map $j: (\mathrm{Def}_Y, *) \to (\mathrm{Def}_X, *)$. For $t \in \mathrm{Def}_Y$, let Y_t be the corresponding fiber of $\mathcal{Y} \to \mathrm{Def}_Y$ (so that $Y = Y_0$). j induces a map $Y_t \to X_{j(t)}$ with the property that $X_{j(t)}$ has a finite number of rational double points, and $Y_t \to X_{j(t)}$ is the minimal resolution of singularities. We define the root system of $X_{j(t)}$ to be the direct sum of the root systems of the individual double points; this is naturally a subset of $H^2(Y_t, \mathbb{Z})$.

<u>Theorem</u> ([4], [5], [26], [32])

There are representatives of the versal deformations $(\mathcal{Y}, Y) \to (\mathrm{Def}_Y, *)$ and $(\mathcal{X}, X) \to (\mathrm{Def}_X, *)$ such that

1) $\mathrm{Def}_Y = \Lambda_{\mathbb{C}}$, $\mathrm{Def}_X = \Lambda_{\mathbb{C}}/W$ (with $* = 0$), and $\mathrm{Def}_Y \to \mathrm{Def}_X$ is the natural quotient by the action of the Weyl group.
2) If $q: \mathcal{Y} \to \Lambda_{\mathbb{C}}$ is the versal family, then there is a trivialization $i: q_*(\mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that for each $t \in \Lambda_{\mathbb{C}} = \mathrm{Def}_Y$,

$$i_t(R(Y_t)) = \{ \delta \in R \mid \langle \delta, t \rangle = 0 \}.$$

<u>Corollary 1</u> ("Simultaneous resolution")

Let $(\mathcal{X}, X) \to (S, *)$ be an arbitrary local deformation of X. Then there is a finite map $T \to S$ such that the induced family $\mathcal{X}_T \to T$ admits a <u>simultaneous resolution</u>, in other words, a diagram

$$\begin{array}{ccc} \mathcal{Y}_T & \longrightarrow & \mathcal{X}_T \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

such that the induced map on fibers $Y_t \to X_t$ is a minimal resolution of singularities for each $t \in T$.

Proof: Define $T \to S$ by means of the base change $\mathrm{Def}_Y \to \mathrm{Def}_X$ as follows:

$$\begin{array}{ccc} T & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Def}_Y & \longrightarrow & \mathrm{Def}_X \end{array}$$

and let $\mathcal{Y}_T$ be the family induced by the map $T \to \mathrm{Def}_Y$. The corollary is now immediate. Q.E.D.

Corollary 2

Let $p\colon \mathcal{Y} \to D$ be a one-parameter family of surfaces, where D is the unit disk. Let $C \subset Y_0 = p^{-1}(0)$ be a smooth rational curve of self-intersection -2 which is disjoint from the singular locus of Y_0. Then either there is a divisor $\mathcal{C} \subset \mathcal{Y}$ with $C_t = \mathcal{C} \cap Y_t$ a smooth rational curve of self-intersection -2 for each $t \in D$, or there is a birational map

$$e\colon \begin{array}{ccc} \mathcal{Y} & \dashrightarrow & \mathcal{Y}' \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

whose indeterminacy locus is C. In the latter case, $Y_0 \cong Y_0'$ (although the isomorphism is not induced by the rational map e) and if $\delta \in H^2(Y_0,\mathbb{Z})$ is the class of C, then e induces a map

$$e^*\colon H^2(Y_0,\mathbb{Z}) \longrightarrow H^2(Y_0,\mathbb{Z})$$

which coincides with the reflection s_δ.

<u>Proof</u>: Consider the contraction $Y_0 \to X_0$ of the curve C on the central fiber. By [29], there is a contraction $\mathcal{Y} \to \mathcal{X}$ which induces this, so that

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

is a simultaneous resolution of singularities of $\mathcal{X}$. X_0 has an ordinary double point, so if we let $\Lambda = \mathbb{Z}C$, then the family $\mathcal{Y} \to D$ determines a map $D \to \Lambda_{\mathbb{C}}$.

If the image of D in $\Lambda_{\mathbb{C}}$ is a point, we get the first alternative: the nearby fibers X_t all have ordinary double points, so that their resolutions Y_t have a family of curves $C_t \subset Y_t$. If the image of D is a curve, then letting the Weyl group $W \cong \mathbb{Z}/2\mathbb{Z}$ act, we get a second map $D \to \Lambda_{\mathbb{C}}$ which induces the famiy $\mathcal{Y}'$; the natural birational map $\mathcal{Y} \dashrightarrow \mathcal{Y}'$ is an isomorphism outside C. The statement about e^* is proved in [6]. Q.E.D.

<u>Definition</u>

The birational map constructed in corollary 2 is called the <u>elementary modification with center C</u>. We remark that the elementary modification also has a description in terms of blowups and blowdowns: cf. [9], [12], [23], [28].

Suppose now that X is a compact surface with rational double points, and $\rho: Y \to X$ be the minimal resolution. We define the <u>root system</u>, (resp. the <u>Weyl group</u>) of X, denoted R(X) (resp. W(X)), to be the direct sum of the root systems (resp. the direct product of the Weyl groups) of the individual double points. Let

$$\delta_1, \ldots, \delta_k \in H^2(Y,\mathbb{Z})$$

be the classes of the irreducible curves which are contracted by ρ. Then

$$R(X) = \{ \delta = \Sigma a_i \delta_i \in H^2(Y,\mathbb{Z}) \mid a_i \in \mathbb{Z} \text{ and } \langle\delta,\delta\rangle = -2 \},$$

and $W(X)$ has a natural representation on $H^2(Y,\mathbb{Z})$ generated by the reflections $\{ s_\delta \mid \delta \in R(X) \}$. We define $I^2(X) = H^2(Y,\mathbb{Z})^{W(X)}$; this is exactly the set of cohomology classes which are orthogonal to the classes in $R(X)$.

<u>Corollary 3</u>

Let $p\colon \mathcal{X} \to S$ be a proper family of surfaces with only rational double points as singularities. There is a sheaf $I^2(\mathcal{X}/S)$ such that for each $s \in S$, if U is a sufficiently small neighborhood of $s \in S$, then

$$\Gamma(U, I^2(\mathcal{X}/S)) \cong H^2(Y_s,\mathbb{Z})^{W(X_s)},$$

where $\rho_s\colon Y_s \to X_s$ is the minimal resolution of X_s and $W(X_s)$ is the Weyl group.

<u>Proof</u>. Let us call a neighborhood U of s <u>small</u> if the germ $(U,s) \to (\mathrm{Def}_{X_s},*)$ has a representative $U \to \mathrm{Def}_{X_s}$. If U is a small neighborhood, we define $\tilde{U} \to U$ by means of the $W(X_s)$ base change:

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Def}_{Y_s} & \longrightarrow & \mathrm{Def}_{X_s} \end{array}$$

and let $\tilde{q}\colon \mathcal{Y}_{\tilde{U}} \to \tilde{U}$ be a simultaneous resolution of $\tilde{p}\colon \mathcal{X}_{\tilde{U}} \to \tilde{U}$. For a small neighborhood, there is a trivialization

$$R^2\tilde{q}_*(\mathbb{Z}) \cong H^2(Y_s,\mathbb{Z}).$$

We will define sections and restriction maps for $I^2(\mathfrak{X}/S)$ over small neighborhoods; a slight modification of the usual sheafification construction then produces the desired sheaf. First define, for every small neighborhood U of s,

$$\Gamma(U,I^2(\mathfrak{X}/S)) \cong H^2(Y_s,\mathbb{Z})^{W(X_s)}.$$

Next suppose that U is a small neighborhood of $s \in S$, V is a small neighborhood of $t \in S$, and $V \subset U$. Let $\tilde{U} \to U$ be the $W(X_s)$ basechange, and pick a point $r \in \tilde{U}$ in the inverse image of t. If $\tilde{q}: \mathcal{Y}_{\tilde{U}} \to \tilde{U}$ is a simultaneous resolution, then a trivialization

$$R^2\tilde{q}_*(\mathbb{Z}) \cong H^2(Y_s,\mathbb{Z})$$

induces an isomorphism

$$i: H^2(Y_s,\mathbb{Z}) \tilde{\to} H^2(Y_r,\mathbb{Z}) = H^2(Y_t,\mathbb{Z}).$$

Thus, to construct a restriction map

$$\Gamma(U,I^2(\mathfrak{X}/S)) \to \Gamma(V,I^2(\mathfrak{X}/S))$$

we only need to verify that

$$i(H^2(Y_s,\mathbb{Z})^{W(X_s)}) \subset H^2(Y_t,\mathbb{Z})^{W(X_t)}$$

and that the induced map is independent of choices. But this is clear, by the idenfication of the root systems of X_s and X_t given in the theorem above. Q.E.D.

Notice that the sheaf $I^2(\mathfrak{X}/S)$ has a natural bilinear pairing to $\mathbb{Z}_S$ induced by the intersection form in the fibers; by Poincaré duality, and the fact that the Weyl group representations preserve the intersection pairing, this is a perfect pairing after tensoring with $\mathbb{Q}$.

In fact, $I^2(\mathfrak{X}/S) \otimes \mathbb{Q}$ admits an alternate description as the <u>relative intersection homology</u> (cf. [11]) of the family $\mathfrak{X} \to S$. This follows from the Brieskorn-Grothendieck description of the simultaneous resolution in group theoretic terms (cf. [5], [32]) together with a result of Borho-MacPherson [1] which characterizes the intersection homology as the W-invariant part of the cohomology of the simultaneous resolution.

<u>Definition</u>

Let $\mathfrak{X} \to S$ be a proper family of surfaces with only rational double points as singularities, let $Y_s \to X_s$ be the minimal resolution of one fiber, and let $L \cong H^2(Y_s,\mathbb{Z})$ be the cohomology lattice. A <u>marking</u> of the family $\mathfrak{X} \to S$ is a metric injection

$$\alpha\colon I^2(\mathfrak{X}/S) \hookrightarrow \underline{L}_S$$

such that for each $t \in S$, $\alpha|_t$ extends to an isometry of $H^2(Y_t,\mathbb{Z})$ with L.

4. Extensions of the period maps

A <u>generalized K3 surface</u> is a compact complex surface X with at worst rational double points, whose minimal resolution $Y \to X$ is a K3 surface. We call the generalized K3 surface <u>Kählerian</u> if its minimal resolution is a Kählerian K3 surface. If X is a Kählerian generalized K3 surface with minimal resolution $Y \to X$, then we have the root system $R(X) \subset H^2(Y,\mathbb{Z})$ and the Weyl group $W(X)$ as in section 3; in addition, we define

$$V_P^+(X) = \{ \kappa \in \overline{V_P^+(Y)} \mid \text{for all } \delta \in \Delta^+(Y),\ \langle\delta,\kappa\rangle = 0 \text{ if and only if } \delta \in R(X) \}.$$

Notice that $V_P^+(X) \subset I^2(X) \otimes \mathbb{R}$, and that this agrees with our earlier definition if $Y = X$.

Since $I^2(X) \subset H^2(Y,\mathbb{Z})$ is the orthogonal complement of a collection of integral (1,1) classes, $I^2(X) \otimes \mathbb{C}$ inherits a Hodge structure from the one on $H^2(Y,\mathbb{C})$. Thus, given a marking $\alpha\colon I^2(X) \hookrightarrow L$, we may put a Hodge structure on $L_{\mathbb{C}}$ by defining $H^{2,0}$ and $H^{0,2}$ to be the corresponding subspaces of $I^2(X) \otimes \mathbb{C}$, and $H^{1,1}$ to be the orgthogonal complement; we thus get a point in the classical period domain Ω corresponding to a marked generalized K3 surface. If $p\colon \mathfrak{X} \to S$ is a family of generalized K3 surfaces with marking $\alpha\colon I^2(\mathfrak{X}/S) \hookrightarrow \underline{L}_S$, the corresponding map $\tau_S\colon S \to \Omega$ is clearly holomorphic.

Suppose now that X is a Kählerian generalized K3 surface, and $\rho\colon Y \to X$ its minimal resolution. A <u>weak polarization</u> on X is a class $\kappa \in V_P^+(X) \subset I^2(X) \otimes \mathbb{R}$ (so that if X is smooth, a weak polarization is just a polarization). If $\mathfrak{X} \to S$ is a marked family of Kählerian generalized K3 surfaces, with marking $\alpha\colon I^2(\mathfrak{X}/S) \to \underline{L}_S$, a <u>weak polarization</u> on $\mathfrak{X} \to S$ is a section

$$\kappa \in \Gamma(S, I^2(\mathfrak{X}/S) \otimes \mathbb{R}) \xrightarrow{\alpha} \Gamma(S, \underline{L}_S \otimes \mathbb{R}) \cong L_{\mathbb{R}}$$

such that $\kappa|_s$ is a weak polarization on X_s for every $s \in S$.

Thus, for a marked, weakly polarized family $\mathfrak{X} \to S$ of Kählerian generalized K3 surfaces, we get the <u>weakly polarized period map</u> $\overline{\tau}_S^p\colon S \to \overline{M}$ by using the classical period map toether with the class $\alpha(\kappa) \in L_{\mathbb{R}}$. This is again a quasi-analytic map, and $\overline{\tau}^p(s) \in \overline{M} - M$ if and only if X_s is singular.

For this period map, we have the

Weakly Polarized Global Torelli Theorem

Let (X,κ) and (X',κ') be Kählerian generalized K3 surfaces with weak polarization, and let $\rho: Y \to X$ and $\rho': Y' \to X'$ be the minimal resolutions. Suppose that $\phi: I^2(X) \to I^2(X')$ is an isometry preserving the Hodge structure such that $\phi(\kappa) = \kappa'$, which extends to an isometry $\psi: H^2(Y,\mathbb{Z}) \to H^2(Y',\mathbb{Z})$. Then there is a unique isomorphism $\Phi: X \tilde{\to} X'$ such that $\Phi^* = \phi$.

Proof: Consider the induced isometry $\tilde{\psi}: I^2(X)^{\perp} \to I^2(X')^{\perp}$. The root system $R(X)$ is characterized as

$$R(X) = \{ \ \delta \in I^2(X)^{\perp} \mid \langle\delta,\delta\rangle = -2 \ \}$$

(and similarly for $R(X')$), so $\tilde{\psi}$ establishes an isomorphism of root systems. Define

$$C(X) = \{ \ v \in I^2(X)^{\perp} \mid \langle v,\delta\rangle > 0 \text{ for every } \delta \in \Delta^+(Y) \cap R(X) \ \}$$

(and similarly for $C(X')$). $\tilde{\psi}(C(X))$ and $C(X')$ are both (open) fundamental domains for the action of $W(X')$ on $I^2(X')^{\perp}$. Thus, there is some $w \in W(X')$ such that $w \circ \psi(C(X)) = C(X')$. Since $\psi|_{I^2(X)} = w \circ \psi|_{I^2(X)}$, by replacing ψ with $w \circ \psi$ we may assume that $\psi(C(X)) = C(X')$.

We now apply the Burns-Rapoport Global Torelli Theorem to ψ. We have $\kappa \in V^+(Y)$, $\kappa' \in V^+(Y')$ and $\psi(\kappa) = \kappa'$, so that $\psi(V^+(Y)) = V^+(Y')$. Since $\psi(C(X)) = C(X')$, if $\delta \in \Delta^+(Y) \cap R(X)$ then $\psi(\delta) \in \Delta^+(Y') \cap R(X')$. Also, if $\delta \in \Delta^+(Y)$, $\delta \notin R(X)$ then $\langle\kappa,\delta\rangle > 0$; hence $\langle\kappa',\psi(\delta)\rangle > 0$ so that $\psi(\delta) \in \Delta^+(Y')$. Thus, $\psi(\Delta^+(Y)) = \Delta^+(Y')$ and the Burns-Rapoport Global Torelli theorem applies: there is a unique isomorphism $\Psi: Y \tilde{\to} Y'$. Since this induces isomorphisms on the exceptional sets of ρ and ρ', Ψ descends to a unique isomorphism $\Phi: X \tilde{\to} X'$. Q.E.D.

Unfortunately, the condition that $\phi: I^2(X) \to I^2(X')$ extend to an isometry $\psi: H^2(Y,\mathbb{Z}) \to H^2(Y',\mathbb{Z})$ cannot be relaxed. In an appendix to this paper, we give an example of two non-isomorphic, weakly polarized Kählerian generalized K3 surfaces (X,κ) and (X',κ') with an isometry $\phi: I^2(X) \to I^2(X')$ preserving the Hodge structure and the polarization; of course, ϕ does not extend to an isometry $\psi: H^2(Y,\mathbb{Z}) \to H^2(Y',\mathbb{Z})$.

We now consider the moduli of weakly polarized, marked, Kählerian generalized K3 surfaces, beginning with local moduli. If X is a generalized K3 surface, there is a local universal deformation $(\mathfrak{X},X) \to S$ of X. Let $P_1, \ldots, P_k$ be the singular points of X, and let $X_{P_1}, \ldots, X_{P_k}$ be the germs of X at its singular points. Since S is smooth, by a theorem of Burns and Wahl [7], there is a map

$$j: S \to \mathrm{Def}_{X_{P_1}} \times \cdots \times \mathrm{Def}_{X_{P_k}}$$

which is a local fibration. In order to get a local universal <u>marked</u> family for X, we form the W(X) base change $\tilde{S} \to S$ (using the map j), and let $\mathfrak{X}_{\tilde{S}} \to \tilde{S}$ be the induced family; this is clearly universal among marked families (by the theorem in section 3). Note that the local Torelli theorem for a simultaneous resolution $\mathfrak{Y}_{\tilde{S}} \to \mathfrak{X}_{\tilde{S}}$ (for the classical period map) gives a local Torelli theorem for the classical period map of the universal marked family as well.

As in section 2, we also need a local marked <u>polarized</u> family. Suppose that $\kappa \in V_P^+(X)$ is a weak polarization, and $\mathfrak{X}_{\tilde{S}} \to \tilde{S}$ is the universal marked family. We consider the hypersurface $T \subset \tilde{S}$ on which the class κ remains of type (1,1). We claim that in fact κ is a weak polarization on all nearby fibers of the induced family $\mathfrak{X}_T \to T$. This is due to the description of the root systems of the nearby fibers given in the theorem in section 3: since for each $t \in T$ (sufficiently close to the origin) we have $R(X_t) \subset R(X_0)$, we get that $\kappa \perp R(X_t)$, which guarantees that $\kappa \in V_P^+(X_t)$ (since κ is of type (1,1)).

To construct a global moduli space, consider the moduli functor which associates to each complex analytic space S the set of isomorphism classes of triples

$$(p\colon \mathfrak{X} \to S,\quad \alpha\colon I^2(\mathfrak{X}/S) \hookrightarrow \underline{L}_S,\quad \kappa \in \Gamma(S, I^2(\mathfrak{X}/S) \otimes \mathbb{R}))$$

of marked families of generalized K3 surfaces with a weak polarization. This functor is again not representable in the category of complex analytic spaces, but we can "represent" it by a subset $\overline{M}_0 \subset \overline{M}$, a family $\mathfrak{X}_{\overline{M}_0} \to \overline{M}_0$ and the quasi-analytic maps $S \to \overline{M}_0$. We construct $\mathfrak{X}_{\overline{M}_0} \to \overline{M}_0$ by the same methods we have used before: for each marked, generalized Kählerian K3 surface X (with weak polarization κ), there is a versal deformation $(\mathfrak{X}, X) \to (T,*)$ of marked , weakly polarized K3 surfaces, and the period map $\overline{\tau}_T^P\colon T \to \overline{M}$ is a local isomorphism onto its image. Using the period maps $\overline{\tau}_T^P$, we may glue together the families $\mathfrak{X} \to T$ to obtain the "moduli space" $\overline{M}_0 \subset \overline{M}$.

The action of the automorphism group Γ of L on $\overline{M}$ is discrete and properly discontinuous. Thus, we may also form the quotient $\overline{M}_0/\Gamma$, which we regard as a "coarse moduli space" of weakly polarized Kählerian generalized K3 surfaces. ($\overline{M}_0/\Gamma$ has no universal family, so this is not a fine moduli space.) Note that if X is a singular generalized K3 surface, $p\colon \mathfrak{X} \to S$ its local universal deformation, and $\tilde{S} \to S$ is the W(X) cover, then the map $\overline{\tau}_{\tilde{S}}^P\colon \tilde{S} \to \overline{M}_0$ descends to a map $\tau\colon S \to \overline{M}_0/\Gamma$ (since the W(X)-action is compatible with the Γ-action). This local moduli map has been studied by Looijenga [16], using other methods.

5. Period maps for algebraic K3 surfaces

Let κ be a weak polarization on a generalized K3 surface X. We call the polarization algebraic if there is some $\ell \in L$ with $\langle \ell, \ell \rangle > 0$ and $\kappa = \ell/\sqrt{\langle \ell, \ell \rangle}$. For each such ℓ, we define the algebraic period domain (of type ℓ) to be

$$\Omega_\ell = \{ x \in \Omega \mid \ell \text{ is type } (1,1) \text{ on } x \}.$$

A marked weakly polarized family $\mathfrak{X} \to S$ of K3 surfaces for which the weak polarization is algebraic on each fiber induces a holomorphic map $S \to \Omega_\ell$.

If we embed $\Omega_\ell \to \Omega \times L_{\mathbb{R}}$ by using the constant class $\kappa = \ell/\sqrt{\langle \ell, \ell \rangle}$ on the second factor, then the image lies in $\overline{M}$; the map $\Omega_\ell \to \overline{M}$ can be regarded as a natural transformation between two moduli functors, in which we "forget" that the weak polarization is algebraic. Notice that the image of Ω_ℓ meets $\overline{M} - M$, so there is no corresponding natural transformation $\Omega_\ell \to M$; in particular, there is no natural transformation $\Omega_\ell \to \tilde{\Omega}$, i.e., the algebraic period domain cannot be naturally embedded in the Burns-Rapoport period domain.

Algebraic polarizations are closely related to ample divisors. Recall that a divisor D on a smooth surface Y is pseudoample if the linear system $|nD|$ gives a birational map for n sufficiently large.

Theorem (Mayer [19])

Let ℓ be the cohomology class of an effective divisor D on a smooth K3 surface Y. Then the linear system $|D|$ is ample (resp. pseudoample) if and only if $\langle \ell, \ell \rangle > 0$ and $\langle \ell, \delta \rangle > 0$ (resp. $\langle \ell, \delta \rangle \geq 0$) for all $\delta \in \Delta^+(Y)$. In the pseudoample case, for n sufficiently large, $|nD|$ contracts all curves $\delta \in \Delta^+$ with $\langle \ell, \delta \rangle = 0$, and embeds the resulting surface X.

In other words, $|D|$ is ample when $<\ell,\ell> > 0$ and $\ell \in V_p^+(Y)$, and $|D|$ is pseudoample when $<\ell,\ell> > 0$ and $\ell \in \overline{V_p^+(Y)}$.

Corollary

Let X be a Kählerian generalized K3 surface. If $\mathcal{L}$ is an ample line bundle on X, then $\mathcal{L}$ has a cohomology class $\ell \in I^2(X)$, and

$$\kappa = \frac{\ell}{\sqrt{<\ell,\ell>}} \in I^2(X) \otimes \mathbb{R}$$

is a weak algebraic polarization on X. Conversely, every weak algebraic polarization corresponds to an ample line bundle on X.

Proof: Let $\rho: Y \to X$ be the minimal resolution. $\rho^* \mathcal{L}$ is pseudoample on Y and has a cohomology class $\ell \in H^2(Y,\mathbb{Z})$. Since $<\ell,\delta> = 0$ for each $\delta \in R(X)$, we have $\ell \in I^2(X)$. The conditions in Mayer's theorem now guarantee that κ is a weak polarization on X, which is algebraic by construction.

Conversely, if κ is a weak algebraic polarization, then the corresponding class ℓ is an integral (1,1) class contained in $\overline{V_p^+(Y)}$ with $<\ell,\ell> > 0$. By Riemann-Roch, $\pm \ell$ is the class of an effective divisor. But $-\ell$ cannot be effective since $\ell \in V^+(Y)$; thus ℓ corresponds to a pseudoample linear system $|D|$, which descends to an ample line bundle on X. Q.E.D.

6. Surjectivity of the period maps

In [14], Kulikov proved the following theorem:

Algebraic Surjectivity I

For every point $x \in \Omega_\ell$ there is a marked K3 surface whose classical periods are x.

Proof sketch: It is well known (cf. [31, Chap. IX]) that the image of the algebraic period map is a dense open subset of Ω_ℓ (in the classical topology). Let D be the unit disk, and $D^* = D - \{0\}$. If $x \in \Omega_\ell$ is not in the image of the period map, there is a map $\psi: D \to \Omega_\ell$ such that $\psi(D^*)$ is in the image, and $\psi(0) = x$. One first shows that there is a projective family of surfaces (not necessarily smooth or with rational double points) $\mathfrak{X} \to D$ such that $\mathfrak{X}^* \to D^*$ is a family of smooth marked K3 surfaces with an algebraic polarization whose period map coincides with $\psi|_{D^*}$. (We may, if we wish, choose ψ in such a way that each fiber X_t ($t \in D^*$) has Picard number 1.) The classification of degenerations ([13], [15], and [24]) guarantees that after a basechange, $\mathfrak{X}$ is birational to a family $\mathfrak{X}' \to D$ of smooth marked K3 surfaces; since Ω is separated, the classical periods of $\mathfrak{X}'_0$ coincide with $x = \psi(0)$.
Q.E.D.

Proposition

Let $\mathfrak{X} \to D$ be a family of (smooth) marked K3 surfaces over the disk such that the restricted family $\mathfrak{X}^* \to D^*$ has an algebraic polarization and such that each fiber X_t ($t \in D^*$) has Picard number 1. Then there is a marked family of generalized K3 surfaces $\mathfrak{X}' \to D$ with a weak algebraic polarization, such that the restricted polarized marked families coincide.

Proof: Let W be the group generated by the reflections

$$\{ s_\delta \mid \delta \in \Delta(X_0) \},$$

and let $\pm W$ be the group generated by W and ± 1 (acting on $H^2(X_0)$).

There is an induced action of $\pm W$ on $V(X_0)$; by a result of Vinberg [35] (see also [23]), $V_P^+(X_0)$ is an (open) fundamental domain for the action of $\pm W$ on $V(X_0)$. Thus, if κ is the weak algebraic polarization on X_t ($t \in D^*$), there is some $w \in W$ with $\pm w(\kappa) \in V_P^+(X_0)$.

We may write $w = s_{\delta_k} \cdots s_{\delta_1}$, where each δ_i is the class of an irreducible curve C_i on X_0. We now perform the corresponding sequence of elementary modifications on $\mathfrak{X}$ (centered at $C_1, \ldots, C_k$): these modifications may be performed since the Picard number of X_t is 1. We arrive at a family $\mathfrak{X}'' \to D$ with the property that κ is a weak polarization on X_t'' for $t \in D^*$, and

$$\pm \kappa \in \overline{V_P^+(X_0)} \subset V^+(X_0).$$

But the period map $D \to \tilde{\Omega}$ induces a <u>continuous</u> family of choices of V^+; thus in fact $\kappa \in V^+(X_0)$, so that $\kappa \in \overline{V_P^+(X_0)}$.

By Mayer's theorem, since $\kappa = \ell/\sqrt{\langle \ell, \ell \rangle}$, ℓ is the class of a pseudoample divisor $|D|$ on X_0''. Let ρ_0: $X_0'' \to X_0'$ be the map defined by $|nD|$; X_0' is a generalized K3 surface on which κ is a weak algebraic polarization. By [29], the contraction ρ_0 is induced by a map ρ: $\mathfrak{X}'' \to \mathfrak{X}'$ which commutes with projection to D; $\mathfrak{X}'$ is the desired family.
Q.E.D.

From this proposition and the proof of "algebraic surjectivity I," we immediately deduce:

<u>Algebraic Surjectivity II</u>

For every point $x \in \Omega_\ell$, there is a marked generalized K3 surface (X, α) whose classical periods are x, such that $\ell \in I^2(X)$ is the cohomology class of an ample divisor on X.

Todorov [33] has used "algebraic surjectiviy II" and Yau's solution [37], [38] to the Calabi conjecture [8] to prove

Burns-Rapoport Surjectivity

For every $x \in \tilde{\Omega}$, there is a marked Kählerian K3 surface whose Burns-Rapoport periods are x.

This has also been proved by Looijenga [17] (cf. also Namikawa [21]). In fact, Looijenga proves a stronger theorem (also announced by Todorov):

Polarized Surjectivity

For every $x \in M$, there is a marked polarized Kähler K3 surface X whose polarized periods are x, such that the cohomology class of the Kähler metric corresponds to the polarization (i.e. the polarization is Kählerian).

From this, we can deduce

Weakly Polarized Surjectivity

For every $x \in \overline{M}$, there is a marked, weakly polarized, Kählerian generalized K3 surface whose weakly polarized periods are x.

Proof: $x \in \overline{M}$ corresponds to a Hodge decomposition

$$L_{\mathbb{C}} \cong H^{2,0}(x) \oplus H^{1,1}(x) \oplus H^{0,2}(x)$$

and a class $\kappa \in H^{1,1}(x) \cap L_{\mathbb{R}}$. Let

$$R(x) = \{ \delta \in H^{1,1}(x) \cap L \mid \langle\delta,\delta\rangle = -2 \text{ and } \langle\kappa,\delta\rangle = 0 \},$$

let $W(x)$ be the Weyl group determined by $R(x)$, and let $C \subset L_{\mathbb{R}}$ be an (open) fundamental domain for the action of $W(x)$ on $L_{\mathbb{R}}$.

If $x \in M$, there is nothing to prove. We may thus assume $x \in \overline{M} - M$; in this case, $\kappa \in \overline{C} - C$. Pick an element $\kappa' \in C \cap H^{1,1}(x)$

which is very close to κ, with $\langle\kappa',\kappa'\rangle = 1$. The Hodge decomposition

$$L_{\mathbb{C}} \cong H^{2,0}(x) \oplus H^{1,1}(x) \oplus H^{0,2}(x)$$

together with the class κ' determines a point y M.

By the polarized surjectivity theorem, there is a marked Kähler K3 surface Y with that Hodge structure, whose Kähler metric has cohomology class κ'. The irreducible curves on Y whose cohomology classes lie in $\Delta^+(Y) \cap R(x)$ can be contracted to rational double points by a map ρ: Y → X; it is easy to see that X is the required surface, with weak polarization κ. Q.E.D.

In terms of the moduli spaces described earlier, these surjectivity theorems imply that $\tilde{\Omega}_0 = \tilde{\Omega}$, $M_0 = M$, $N_0 = N$, $\overline{M}_0 = \overline{M}$, and $\overline{M}_0/\Gamma = \overline{M}/\Gamma$.

The weakly polarized global Torelli and surjectivity theorems above are not in their sharpest possible form. Ideally, each weak polarization should be the cohomology class of a Kähler form (in the sense of Moishezon [20] and Fujiki [10]) on X, and the surjectivity theorem should be stated in terms of weak Kähler polarizations. However, the proof would require an analogue of Yau's theorem for these forms -- that every Kähler form is cohomologous to a Ricci-flat Kähler form -- which is not currently available.

Appendix

A lattice is a free $\mathbb{Z}$-module of finite rank with a bilinear form. If R is a (negative definite) root system, we let R also denote the lattice it generates. In particular, A_k, D_k, and E_k will denote the lattices corresponding to those standard root systems. L denotes the K3 lattice, as before.

Proposition

There is a lattice Λ of signature (3,7) which has two primitive embeddings $\phi_i: \Lambda \to L$ (i = 1, 2) such that

$$\phi_1(\Lambda)^{\perp} \cong D_{12}$$

$$\phi_2(\Lambda)^{\perp} \cong D_4 \oplus E_8.$$

Furthermore, Λ is unique up to isomorphism.

Proof: We first compute the finite abelian group $G_k = D_k^*/D_k$. In the notation of Pinkham [27], $D_k = T_{2,2,k-2}$. Lemma 1 of [27] then shows that G_k is generated by 3 elements a, b, c with the relations

$$a + b + c = 2a = 2b = (k-2)c.$$

Elementary algebra now shows that

$$G_k \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if k is odd} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if k is even.} \end{cases}$$

In particular,

$$D_{12}^*/D_{12} \cong (D_4 \oplus E_8)^*/(D_4 \oplus E_8) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

(since E_8 is unimodular).

We now apply the work of Nikulin [22]. In Nikulin's terminology, we have shown above that D_{12} and $D_4 \oplus E_8$ have the same discriminant-form q, and $\ell(A_q)$, which is the minimum number of generators of G_{12} (or G_4), is 2. By corollary 1.12.3 of [22], since

$$rk(L) - rk(D_{12}) = rk(L) - rk(D_4 \oplus E_8) = 10 \geq 2 = \ell(A_q),$$

there are primitive embeddings of D_{12} and $D_4 \oplus E_8$ into L. Let Λ_1 and Λ_2 be the orthogonal complements. Each Λ_i is a lattice of signature (3,7) and discriminant form (-q). But by corollary 1.13.3 of [22], since

$$rk(\Lambda_i) = 10 \geq 4 = 2 + \ell(A_q),$$

there is a unique isomorphism class of such lattices. Thus, $\Lambda_1 \cong \Lambda_2$; we define this to be Λ. Q.E.D.

Corollary

There exist two weakly polarized Kählerian generalized K3 surfaces (X_1,κ_1) and (X_2,κ_2) with an isometry ϕ: $I^2(X_1) \to I^2(X_2)$ preserving the Hodge structure and the polarization, such that X_1 has a unique singular point (of type D_{12}), while X_2 has two singular points (of types D_4 and E_8). In particular, X_1 and X_2 are not isomorphic.

Proof: Choose a Hodge decomposition on Λ

$$\Lambda_{\mathbb{C}} \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

such that $H^{1,1} \cap \Lambda = \{0\}$, and choose $\kappa \in H^{1,1} \cap \Lambda_{\mathbb{R}}$ such that $\langle\kappa,\kappa\rangle = 1$. Let ω be a generator of $H^{2,0}$. For each i = 1, 2, the embedding ϕ_i: $\Lambda \to L$ now induces a Hodge structure on L, given by the point $\omega_i = \phi(\omega_i) \in \Omega$. (For this Hodge structure, $\phi_i(\Lambda)^\perp$ is purely of type (1,1).) If we set $\kappa_i = \phi_i(\kappa)$, we get a point $(\omega_i,\kappa_i) \in \overline{M}$.

By the weakly polarized surjectivity theorem, there is a marked, weakly polarized, Kählerian generalized K3 surface X_i whose weakly polarized periods are (ω_i,κ_i). By construction, $R(X_i) = \phi_i(\Lambda)^\perp$ and $I^2(X_i) = \phi_i(\Lambda)$. Thus, $\phi = \phi_2 \circ \phi_1^{-1}$ gives an isometry preserving the Hodge structure and the polarization.

Moreover, the singularities of X_i are given by the root system $R(X_i) = \phi_i(\Lambda)^\perp$. By construction of ϕ_i, we thus see that X_1 has a D_{12}

singularity, while X_2 has a D_4 singularity and an E_8 singularity; X_1 and X_2 cannot be isomorphic. Q.E.D.

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