# On the Néron models of abelian surfaces with quaternionic multiplication

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## Introduction

In this paper we construct a certain proper regular model for abelian surfaces with quaternionic multiplication. The construction applies when the residue characteristic $p > 3$ or more generally in the *tamely ramified* case. Our model has the property that the Néron model can be recovered as its smooth locus. We therefore obtain both the Néron model and a compactification of it. We are able to give a Kodaira-like classification of the special fibers in these proper regular models. We also list the possible groups of connected components in the Néron model.

To set these results in context it is necessary to review the known results on models of elliptic curves and the situation for abelian varieties in higher dimensions. Originally Kodaira classified minimal models of complex analytic degenerations of elliptic curves over the unit disk in $\mathbb{C}$. Minimal here means proper and regular such that no irreducible components of the special fiber can be contracted without losing regularity of the total space. Minimal models for complex elliptic surfaces always exist and are unique. Kodaira organized the possible singular fibers into a classification scheme which has been the prevailing notation ever since.

Néron considered elliptic curves in the final section of his seminal paper [15] in which the existence of Néron models for abelian varieties is established. He classified the minimal models for elliptic curves over discrete valuation rings, recovering a classification identical to the earlier one of Kodaira in the complex analytic case. This classification also appears in Tate's algorithm [17]. In this algorithm Tate essentially shows how to resolve the singularities in the plane model of the elliptic curve provided by the generalized Weierstraß equation to arrive at the minimal model. The Néron model of an elliptic curve is the smooth locus of its minimal model, cf. [2], Proposition 1.5.

When one turns to higher dimensional abelian varieties this procedure for dimension 1 does not generalize at all. The first difficulty is with the notion of "minimal model". Even in the complex analytic case, minimal models as defined above are far from unique. The successful study of birational geometry in complex dimension

2 based on these models cannot be extended to higher dimension. However, in the last 10 years Mori and others have found a reformulation of the notion of "minimal model" which leads to a good theory of birational geometry in higher dimension. Mori's minimal models are proper but not necessarily regular. (See [14] and [20] for introductory accounts which contain further references.) Unfortunately, as we remark in section 5 below, Mori's minimal models of abelian varieties are not related to Néron models in any nice way.

A second difficulty in higher dimension is that it is not even known whether proper regular models exist, although it is surely conjectured that they do. The problem is that the mixed characteristic resolution of singularities is not known in sufficient generality to deduce that a proper regular model can always be obtained by blowing up. Even when proper regular models exist they are obviously not in any sense unique. So what is required is some condition on proper regular models (other than "minimality") which while perhaps not making such models unique will at any rate allow recovery of the Néron model. Ideally, the Néron model could be recovered as the smooth locus of such a model.

In this paper we consider these problems for the special case of abelian varieties with potential good reduction. Here one at least has a very natural model with which to begin. Namely pass to a finite extension with Galois group $G$ where good reduction is obtained. Consider the Néron model over this finite extension. By the universal property $G$ acts on this Néron model, the induced automorphisms of the special fiber being geometric. Now take the quotient by $G$ to obtain a proper model for the abelian variety over the original discrete valuation ring. This model frequently has singularities which are amenable to analysis. For example, if the singularities are isolated and if the residue characteristic is prime to $|G|$, then only cyclic quotient singularities arise. However even in the cyclic case it does not seem to be known that the singularities can always be resolved to yield a proper regular model. But in two cases we are able to explicitly resolve the singularities—namely elliptic curves with integral $j$-invariant and abelian surfaces with quaternionic multiplication. These computations comprise section 3 of this paper. Hence in these cases we can blow up to obtain a proper regular model.

But of course more is wanted. We isolate the general notion of a proper regular model being a *good model* in section 1 to play an analogous rôle to that of minimal models for elliptic curves—its defining properties ensure that its smooth locus furnishes a Néron model. Good models are not known to exist in general and are not unique. However for elliptic curves with potential good reduction and abelian surfaces with quaternionic multiplication we show that the proper regular models produced by our explicit resolution of singularities are good models. Hence here we are able to geometrically construct the Néron models by this method. The results for elliptic curves rather reassuringly yield the known classification of Néron. However it is very amusing to see this classification emerge without ever appealing to an equation. In the case of abelian surfaces with quaternionic multiplication the results not surprisingly look like the "square" of the results for elliptic curves with potential good reduction.

Our construction is motivated by the analysis of complex analytic degenerations of abelian surfaces over the unit disk in $\mathbb{C}$ given by Ueno [18] in the case of finite monodromy. Minimal models for those degenerations (in the sense of Mori) have been constructed by Crauder and the second author [5].

As this work neared completion, we became aware of the related recent work of Edixhoven [6]. Edixhoven's beautiful characterization of the Néron model can be applied to the cases we consider of elliptic curves with integral $j$-invariant and abelian surfaces with quaternionic multiplication. In particular, another proof of the classification of the possible groups of connected components (Table 1 in section 4 below) can be given by combining our Theorem 4.3 with [6], Theorem 5.3.

It remains for us to discharge the agreeable duty of acknowledging our reliance on the expositions of Néron's work in more modern language given by Artin [1] and Bosch-Lütkebohmert-Raynaud [2]. These texts are cited throughout. Also we would like to thank Ching-Li Chai, Gerd Faltings, and Robert Friedman for very helpful discussions. This work was partially supported through NSA and PSC-CUNY grants to the first author and an NSF grant to the second author.

# 1 Proper regular models and Néron models of abelian varieties

Let $R$ be a discrete valuation ring with fraction field $K$, uniformizer $\pi$, and perfect residue field $k$. We describe in this section how the Néron model of an abelian variety $X_K$ over $K$ can be computed from a proper regular $R$-model $X$ of $X_K$. Proper regular $R$-models are not known to exist in general, but we will explicitly construct proper regular $R$-models for abelian surfaces with quaternionic multipliction and for integral $j$-invariant elliptic curves later in the paper.

Let $X$ be a regular $R$-scheme. Such a scheme is not generally smooth over $R$; we let $X^{\mathrm{sm}} \subseteq X$ denote the complement of the nonsmooth locus. Each irreducible component $E_i$ of the special fiber $X_k$ has a *multiplicity* in the fiber

$$\mathrm{mult}(E_i) = v_{E_i}(\pi).$$

We then have

$$X^{\mathrm{sm}} \subseteq X - \bigcup_{\mathrm{mult}(E_i)\neq 1} E_i.$$

If $E_i$ is a component of multiplicity 1, then $X$ is generically smooth along $E_i$. In other words, if $\eta_i$ denotes the generic point of $E_i$, there is an open subscheme $X_i \subseteq X^{\mathrm{sm}}$ such that the special fiber $(X_i)_k$ is irreducible with generic point $\eta_i$.

**Definition 1.1** Let $X_K$ be a smooth variety over $K$ of dimension $d$, let $X_i$ be a smooth $R$-model of $X_K$ with irreducible special fiber, and let $\omega$ be a nonzero differential form of degree $d$ on $X_K$. The *order of* $\omega$ *at* $X_i$ is the integer $n$ such that $\pi^{-n}\omega$ extends to a generator of $\Omega^d_{X_i/R}$ at the generic point $\eta_i$ of the special fiber $(X_i)_k$. We denote this order by $\mathrm{ord}_{X_i}(\omega)$.

**Definition 1.2** Let $X_K$ be an abelian variety of dimension $d$. Let $\omega$ be a nonzero left-invariant differential form of degree $d$ on $X_K$. A smooth $R$-model $X_i$ of $X_K$ with irreducible special fiber is called $\Omega$-*minimal* if $\mathrm{ord}_{X_j}(\omega) \geq \mathrm{ord}_{X_i}(\omega)$ for all other smooth $R$-models $X_j$ of $X_K$ with irreducible special fiber.

Note that since any other left-invariant differential form $\omega'$ of degree $d$ can be written as $\omega' = k\omega$ for some $k \in K$, the property of being $\Omega$-minimal does not depend on the choice of $\omega$.

If an abelian variety $X_K$ has a proper regular $R$-model $X$, the $\Omega$-minimality condition is easy to check.

**Proposition 1.3** *Suppose $X$ is a proper regular $R$-model of an abelian variety $X_K$. Let $\{E_i\}_{i\in I}$ be the set of components of the special fiber $X_k$ of multiplicity 1, and choose $X_i \subseteq X^{\mathrm{sm}}$ such that $(X_i)_k$ and $E_i$ have the same generic point. Then for any smooth $R$-model $Y$, $Y$ is $\Omega$-minimal if and only if*

$$\mathrm{ord}_Y(\omega) = \min_{i\in I}\{\mathrm{ord}_{X_i}(\omega)\}.$$

We can now state the result on computing the Néron model. We denote the Néron model of $X$ by $X^{\text{Nér}}$.

**Theorem 1.4** *Let $X$ be a proper regular $R$-model and let $\omega$ be a left-invariant differential form of top degree on $X_K$. Let*

$$J = \{j \in I \mid \mathrm{ord}_{X_j}(\omega) = \min_{i\in I}\{\mathrm{ord}_{X_i}(\omega)\}\},$$

*and let*

$$X^{\mathrm{mo}} = X^{\mathrm{sm}} - \bigcup_{i\notin J} E_i.$$

*($X^{\mathrm{mo}}$ omits the components which are not of* minimal order *in $X^{\mathrm{sm}}$.) Then*

1. *The natural map $X^{\mathrm{mo}} \to X^{\text{Nér}}$ is an open immersion which establishes a one-to-one correspondence between components of special fibers.*

2. *If $J = I$, i.e., if $\mathrm{ord}_{X_i}(\omega)$ is constant, then $X^{\mathrm{mo}} = X^{\mathrm{sm}} = X^{\text{Nér}}$.*

*(In general, $X^{\text{Nér}}$ will be the union of a finite number of translates of $X^{\mathrm{mo}}$.)*

In the course of the proof, we will use the following version of Hensel's lemma. The proof given in [13] (which is based on EGA IV [10], Theorem 17.5.1) works equally well in the present context.

**Lemma 1.5** *Let $R$ be a complete discrete valuation ring, $K$ its fraction field, $\mathbf{m}$ its maximal ideal, and $k = R/\mathbf{m}$ its residue field. Let $S = \operatorname{Spec} R$ and suppose $f : X \to S$ is a flat morphism with $X$ a regular scheme. Consider the specialization map*

$$\alpha : \mathrm{Hom}_S(S, X) \to \mathrm{Hom}_S(\operatorname{Spec} k, X_k) = X_k(k).$$

1. *If $X$ is smooth, then $\alpha$ is surjective.*

2. *If $X$ is proper over $S$, let $\beta : \mathrm{Hom}(\operatorname{Spec} K, X_K) \to \mathrm{Hom}_S(S, X)$ be the natural isomorphism. Then the image of $\alpha \circ \beta$ lies in $(X^{\mathrm{sm}})_k(k)$.*

We begin the proof of Theorem 1.4 with a proposition. Let $R^{sh}$ denote the strict henselization of the discrete valuation ring $R$.

**Proposition 1.6** *Let $X$ be an $R$-scheme of finite type whose generic fiber $X_K$ is smooth over $K$, and let $X' \to X$ be a resolution of singularities. (That is, $X' \to X$ is a proper morphism from a regular scheme $X'$ to $X$ which is an isomorphism over the regular locus of $X$.) Let $X'' = X'^{\mathrm{sm}}$. Then $X'' \to X$ is an isomorphism on generic fibers, and $X''(R^{sh}) \to X(R^{sh})$ is bijective.*

This proposition is proved in [2], pp. 61–62. Using it, we can calculate a "weak Néron model" as defined in [2], p. 74.

**Corollary 1.7** *Suppose that $X_K$ admits an $R$-model which is proper over $R$, separated, and which has a resolution of singularities $X' \to X$. Let $X'' = X'^{\mathrm{sm}}$ be the smooth part of $X'$. Then $X'' \to \operatorname{Spec} R$ is a weak Néron model of $X_K$.*

*Proof:* By the valuative criterion for properness, every $K^{sh}$-point of $X_K$ is induced by a $R^{sh}$-point of $X$. Thus, $X(R^{sh}) \to X_K(K^{sh})$ is bijective and so is $X''(R^{sh}) \to X_K(K^{sh})$. The corollary now follows from [2], p. 74. Q.E.D.

*Proof of Theorem 1.4:* Let $\omega$ be a left-invariant differential of degree $d$ on $X_K$, and let $n$ be the common order of $\omega$ at models $X_i \subset X$ with irreducible special fiber.

By Corollary 1.7, $X^{\mathrm{sm}}$ is a weak Néron model. Moreover, the proof of [2], Prop. 2, p. 105, shows that the minimum value of $\{\mathrm{ord}_{X_j} \omega\}$ is achieved on components of the special fiber of $X^{\mathrm{sm}}$; that minimum is therefore $n$.

As explained in [2], Cor. 4, p. 110, the construction of a Néron model from a weak Néron model proceeds as follows. Let $X^{\mathrm{mo}}$ be the union of $\Omega$-minimal models coming from the weak Néron model $X^{\mathrm{sm}}$. Then the natural map $X^{\mathrm{mo}} \to X^{\mathrm{Nér}}$ is an open immersion, whose image is a dense open subscheme. Artin [1], pp. 224–225, points out that the set of components is correct; this also follows from [2], Prop. 4, p. 106.

To prove the second part, note that $J = I$ implies that $X^{\mathrm{mo}} = X^{\mathrm{sm}}$. We must therefore show that the open immersion $X^{\mathrm{sm}} \to X^{\mathrm{Nér}}$ is an isomorphism. Now an open immersion is an isomorphism onto its image, which is a Zariski-open set. Let $Z$ be the complement of the image; we must show that $Z$ is empty. Notice that $Z$ must be a subscheme of the closed fiber $(X^{\mathrm{Nér}})_k$, since the map $X^{\mathrm{sm}} \to X^{\mathrm{Nér}}$ is an isomorphism on generic fibers by construction.

To show that $Z$ is empty, it suffices to pass to the strict henselization $R^{sh}$ (denoting base changes by superscript $sh$). Since the residue field $k^{sh}$ is algebraically closed, if $Z^{sh}$ is nonempty it must have a closed point $x \in (X^{\mathrm{Nér}})^{sh}_{k^{sh}}$. We may apply Hensel's Lemma 1.5 since $R^{sh}$ is complete. By part (1) of Hensel's lemma, since $(X^{\mathrm{Nér}})^{sh}$ is smooth there is an $R^{sh}$-valued point $\xi$ which specializes to $x$. Then $\xi|_{X_K}$ gives a $K$-valued point. Since $X^{sh} \to R^{sh}$ is proper, by part (2) of Hensel's lemma, the specialization $y = (\alpha \circ \beta)(\xi|_{X_K})$ of that point lies in $(X^{sh})^{\mathrm{sm}}$. But then $y$ maps to $x$ under $(X^{\mathrm{sm}})^{sh} \to (X^{\mathrm{Nér}})^{sh}$, contradicting the choice of $x$ as not being in the image of that map. Q.E.D.

**Definition 1.8** Let $X_K$ be an abelian variety of dimension $d$. $X$ is a *good model* for $X_K$ if

1. $X$ is a regular scheme, proper over $\operatorname{Spec} R$, with generic fiber $X_K$, and
2. for any nonzero left-invariant differential $\omega$ of degree $d$ on $X_K$, $\mathrm{ord}_{X_i}(\omega)$ is constant as $X_i \subseteq X$ varies over all smooth $R$-models of $X_K$ with irreducible special fiber.

Using this definition, the second part of Theorem 1.4 can be stated as follows: if $X$ is a good model for $X_K$, then the Néron model of $X_K$ is $X^{\mathrm{sm}}$. In the case of elliptic curves, this result essentially goes back to Néron [15] (cf. also [1], Prop. 1.15).

## 2 The potential good reduction model

In this section we construct models of abelian varieties with potential good reduction over discrete valuation rings. Subsequently in sections 3 and 5 we will resolve the singularities of these good models in several examples to obtain proper regular $R$-models which are *good* in the sense of Definition 1.8. It seems likely that this procedure will always yield a good (proper regular) $R$-model for an abelian variety with potential good reduction. However the resolution of singularities in sufficient generality to conclude this does not seem to be known.

We will work in the strictly local case. To fix notation, let $R$ be a strictly henselian discrete valuation ring with fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of finite characteristic $p$. Denote by $K_s$ a separable closure of $K$.

Let $A/K$ be an abelian variety of dimension $d$ with potential good reduction. Suppose $\mathcal{M}$ is a ring with an embedding $\mathcal{M} \hookrightarrow \mathrm{End}_K(A)$. For a prime $\ell$, the action of $\mathrm{Gal}(K_s/K)$ on the $\ell$-adic Tate module $\mathrm{Ta}_\ell(A) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A(K_s))$ commutes with that of $\mathcal{M}$. If $\ell \neq p$, the representation

$$\mathrm{Gal}(K_s/K) \longrightarrow \mathrm{Aut}_{\mathcal{M}}(\mathrm{Ta}_\ell(A))$$

has image a finite group $\Phi(A/K)$ which is independent of $\ell$ ([16], Theorem 2). Let $K'/K$ be the finite extension cut out by this representation. Then $G = \mathrm{Gal}(K'/K)$ is isomorphic to $\Phi(A/K)$. Set $A' = A \times_K K'$. The group $G$ acts on $A'$ via the second factor and the ring $\mathcal{M}$ acts on $A'$ via the first factor. By the criterion of Néron-Ogg-Shafarevich, $A'$ has good reduction. Let $R'$ be the integral closure of $R$ in $K'$ and set $Y = Y/R'$ equal to the Néron model of $A'$ over $R'$. The special fiber $Y_0 = Y \times_{R'} k$ is then an abelian variety over $k$. By the functoriality of the Néron model the actions of both $G$ and $\mathcal{M}$ on $A'$ extend to the scheme $Y/R'$. Moreover the map $Y \to \mathrm{Spec}\, R'$ is equivariant with respect to the action of $G$, and $G$ acts on $Y_0/k$ via geometric automorphisms. In fact there are canonical isomorphisms

$$\mathrm{Ta}_\ell(A) \cong \mathrm{Ta}_\ell(A') \cong \mathrm{Ta}_\ell(Y_0), \quad \ell \neq p,$$

with the representation $\rho$ of $G$ on $\mathrm{Ta}_\ell(A)$ factoring as follows:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(Y_0/k) \\ \rho\downarrow & & \downarrow \\ \mathrm{Aut}(\mathrm{Ta}_\ell(A)) & \cong & \mathrm{Aut}(\mathrm{Ta}_\ell(Y_0)). \end{array}$$

Further refinements are possible for specific choices of prescribed endomorphisms $\mathcal{M} \hookrightarrow \mathrm{End}_K(A)$. This is taken up in section 4.

The finite group $G$ acts on $Y/R$. We now study the quotient by $G$. The reader is referred to SGA 1 [11], pp. 105–106, for the basic results on taking quotients of schemes by finite groups.

**Proposition 2.1** *The quotient $Y/G = Z$ exists in the category of schemes over $R$.*

*Proof:* The quotient $Z = Y/G$ exists if the action of $G$ on $Y$ is *admissible* (SGA 1 [11], Définition 1.7) in the sense that every orbit of $G$ in $Y$ is contained in a $G$-invariant affine open subset. By [7], Remark 1.10a, p. 7, the abelian scheme $Y/R'$ is projective over $\mathrm{Spec}\, R'$. But $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is finite and hence projective (EGA II [9], Corollaire 6.1.11). Hence the composition $Y \to \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is projective (EGA II [9], Proposition 5.5.5(ii)). But on a scheme projective over an affine scheme $\operatorname{Spec} R$ any finite set of points is contained in an $R$-affine open subset. This is equivalent to the statement that in the ring $R[x_0, ..., x_n]$, $n \geq 1$, the union of finitely many homogeneous prime ideals cannot contain every homogeneous element of positive degree (cf. [3], Chapter II, Proposition 2).

Let $\Gamma$ be an orbit of $G$ in $Y$ with $U$ an $R$-affine open subset containing the finite set $\Gamma$. Consider $U' = \bigcap_{g \in G} gU$. The $G$-invariant open subset $U'$ is $R$-affine since $Y/R$ is separated. As $\Gamma \subseteq U'$, this verifies that the action of $G$ on $Y$ is admissible, concluding the proof. Q.E.D.

**Proposition 2.2**

1. *The generic fiber of $Z = Y/G$ is isomorphic to $A/K$.*
2. *$Z = Y/G \to \operatorname{Spec} R$ is proper.*

*Proof:* The generic fiber of $Z$ is equal to

$$Z \times_R K = (Y/G) \times_R K = (Y \times_R K)/G$$

since $K$ is flat over $R$, cf. SGA 1 [11], Proposition 1.9. But $R' \otimes_R K \approx K'$ and hence

$$(Y \times_R K) = Y \times_{R'} R' \times_R K = Y \times_{R'} K'.$$

Deduce then that

$$Z \times_R K = (Y \times_{R'} K')/G = (A \times K')/G = A,$$

i.e., the generic fiber of $Z$ is $A/K$.

For (2), consider the composition

$$Y \xrightarrow{f} Z \xrightarrow{g} \operatorname{Spec} R.$$

The morphism $f$ is finite and $g \circ f$ is projective (by the proof of Proposition 2.1 above), hence proper. Clearly if $g \circ f$ is of finite type and $f$ is finite, then $g$ is of finite type. We have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_Y} & Y \times_R Y \\ f\downarrow & & \downarrow f \times f \\ Z & \xrightarrow{\Delta_Z} & Z \times_R Z \end{array}$$

in which the horizontal arrows are the diagonal morphisms. Since $Y/R$ is separated, the diagonal $\Delta_Y(Y) \subseteq Y \times_R Y$ is closed. But $f$ (and therefore $f \times f$) is finite and therefore closed. Hence $\Delta_Z(Z) = (f \times f)(\Delta_Y(Y))$ is closed and $Z/R$ is separated. To show that $Z/R$ is proper it remains to show that it is universally closed. This follows from EGA II [9], Corollaire 5.4.3. Q.E.D.

**Definition 2.3** Let $K$ be a local field with ring of integers $R$. Suppose $A/K$ is an abelian variety with potential good reduction. Then the proper model $Z/R$ constructed as in Proposition 2.2 is the *potential good reduction model* of $A/K$.

We now wish to resolve the singularities of $Z$, all of which arise from the points with nontrivial stabilizer. For our purposes, it will suffice to resolve them formally. Henceforth we assume that the order $|G|$ of $G$ is prime to $p$. Then $G$ is a quotient of the tame inertia group of $K$ and hence is cyclic. Moreover if $m = |G|$ is prime to $p$ then the $m^{\text{th}}$ roots of unity are contained in $R$. Also by Kummer theory we can choose a uniformizing parameter $\pi$ for $R$ such that $R'$ is identified with $R[z]/(z^m - \pi)$. Let $\chi_0 : G \to \mu_m \subseteq R^\times$ be the character defined by $\chi_0(g) = g(\sqrt[m]{\pi})/\sqrt[m]{\pi} \in \mu_m$. Note that $\chi_0$ is independent of the choice of uniformizing parameter $\pi$ for $R$ such that $R' \cong R[z]/(z^m - \pi)$.

**Lemma 2.4** *Let $R' = R[z]/(z^m - \pi)$, and let $G = \mathrm{Gal}(R'/R) \cong \mathbb{Z}/m\mathbb{Z}$. The group $G$ acts on $\operatorname{Spec} R'$ through a natural character $\chi_0$: we have $g^*(z) = \chi_0(g)z$.*

*Let $Y$ be a smooth $R'$-scheme of dimension $d$ on which $G$ acts in an $R$-linear manner such that for each $g \in G$, the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \xrightarrow{g} & \operatorname{Spec} R' \end{array}$$

*commutes. Let $P \in Y(k)$ be an isolated fixed point of the action whose stabilizer is $G$, let $\mathbf{m}_P$ and $\overline{\mathbf{m}}_P$ be the maximal ideals of $Y$ and $Y_k$, respectively, at $P$, and let $\widehat{Y}$ be the completion of $Y$ along $P$. Then:*

1. *The action of $G|_{Y_k}$ on the tangent space $\overline{\mathbf{m}}_P/\overline{\mathbf{m}}_P^2$ can be diagonalized. That is, there exists a basis $\eta_1, \ldots, \eta_d$ for $\overline{\mathbf{m}}_P/\overline{\mathbf{m}}_P^2$ as a $k$-vector space, and there exist characters $\bar{\chi}_i : G \to k^\times$ such that $g^*(\eta_i) = \bar{\chi}_i(g)\eta_i$.*

2. *There are formal parameters $y_1, \ldots, y_d$ such that*

$$\widehat{Y} = \operatorname{Spf} R[[y_1, \ldots, y_d]][z]/(z^m - \pi)$$

*and such that the action of $g \in G$ on $\widehat{Y}$ is given by*

$$\begin{aligned} g^*(y_i) &= \chi_i(g)y_i \\ g^*(z) &= \chi_0(g)z \end{aligned}$$

*where $\chi_i : G \to R^\times$ is the natural lift of the character $\bar{\chi}_i$, $i = 1, \ldots, n$.*

*Proof:* Let $y_1^{(1)}, \ldots, y_d^{(1)}$ be any system of formal parameters such that

$$\widehat{Y} = \mathrm{Spf}\, R[[y_1^{(1)}, \ldots, y_d^{(1)}]][z]/(z^m - \pi).$$

By abuse of notation, we let $\mathbf{m}_P$ also denote the maximal ideal of $\widehat{Y}$. We first consider the action of $G$ modulo $\mathbf{m}_P^2$: for each $g \in G$ there is some matrix $A(g) = (a_{ij}(g))$ such that

$$g^*(y_i^{(1)}) \equiv \sum a_{ij}(g) y_j^{(1)} \mod \mathbf{m}_P^2.$$

Since the $m^{\text{th}}$ roots of unity lie in $k$ and $p$ does not divide the order of $G$, the representation of $G$ on the $k$-vector space $\mathbf{m}_P/\mathbf{m}_P^2$ is semisimple. We can therefore diagonalize this action, and by a linear change of variables find formal parameters $y_1^{(2)}, \ldots, y_d^{(2)}$ and characters $\chi_i(g)$ such that

$$g^*(y_i^{(2)}) \equiv \chi_i(g) y_i^{(2)} \mod \mathbf{m}_P^2.$$

The parameters $y_1^{(2)}, \ldots, y_d^{(2)}$ will induce a basis $\eta_1, \ldots, \eta_d$ of $\overline{\mathbf{m}}_P/\overline{\mathbf{m}}_P^2$ on which $g^*$ acts by the restricted characters $\bar{\chi}_i := \chi_i|_{k^\times}$.

Since $m$ is invertible in $R$, we can now define

$$y_i := \frac{1}{m} \sum_{g \in G} \chi_i(g)^{-1} g^*(y_i^{(2)}).$$

Computing this modulo $\mathbf{m}_P^2$ we find

$$\begin{aligned} y_i &\equiv \frac{1}{m} \sum_{g \in G} \chi_i(g)^{-1} \chi_i(g) y_i^{(2)} \mod \mathbf{m}_P^2 \\ &\equiv \frac{1}{m} \sum_{g \in G} y_i^{(2)} \mod \mathbf{m}_P^2 \\ &\equiv y_i^{(2)} \mod \mathbf{m}_P^2 \end{aligned}$$

since $G$ has order $m$. Thus, $y_1, \ldots, y_d$ is a set of formal parameters; moreover, $g^*$ maps $y_i$ to $\chi_i(g) y_i$. Q.E.D.

**Definition 2.5** Let $Z$ be an $R$-scheme whose generic fiber $Z_K$ is smooth and irreducible. A *resolution of singularities* is a proper morphism $f : X \to Z$ of $R$-schemes which induces an isomorphism $f_K : X_K \to Z_K$ on generic fibers such that $X$ is regular. A resolution of singularities *satisfies property* (*) *at* $P$ if there exists a nonzero differential form $\omega$ of degree $d = \dim X_K$ on $X_K$ and an integer $n$ such that $\mathrm{ord}_{E_i} = n$ for all components $E_i$ of $f^{-1}(P)$ with $\mathrm{mult}(E_i) = 1$. The resolution *satisfies property* (**) *at* $P$ if in addition there is only a single component $E$ of $f^{-1}(P)$ with $\mathrm{mult}(E) = 1$.

**Theorem 2.6** *Let $A$ be an abelian variety over $K$, let $R' \to R$ be a finite extension such that $A' = A \times_K K'$ has good reduction. Let $Y \to \operatorname{Spec} R'$ be the Néron model of $A'$. The Galois group $G$ acts on $Y$ (giving automorphisms of $Y$ as an $R$-scheme); let $Z := Y/G$, which is an $R$-scheme. Let $\varphi : Y \to Z$ be the quotient map.*

*We suppose that $R' = R[z]/(z^m - \pi)$, as above. Let $\chi_0$ be the natural primitive character of $G$ determined by the presentation $R' = R[z]/(z^m - \pi)$.*

*Suppose that the origin $e \in Y(k)$ is an isolated fixed point for the action of $G$, and let $\{\chi_1, \ldots, \chi_d\}$ be the characters of $G$ determined by Lemma 2.4.*

*Consider the action of $G$ on $\mathbb{A}^d_{R'} = \operatorname{Spec} R[y_1, \ldots, y_d][z]/(z^m - \pi)$ given by*

$$\begin{aligned} g^*(y_i) &= \chi_i(g) y_i \\ g^*(z) &= \chi_0(g) z. \end{aligned}$$

*Suppose that there is a resolution of singularities $\psi : A_G \to \mathbb{A}^d_{R'}/G$ which satisfies property (*) at $0 \in \mathbb{A}^d_{R'}/G$, and that for every subgroup $H \subseteq G$ there is a resolution of singularities $A_H \to \mathbb{A}^d_{R'}/H$. Then there is a resolution of singularities $f : X \to Z$ such that $X$ is a good model of $A$. (It follows that $X^{\mathrm{sm}}$ is the Néron model of $A$.) Moreover, if $\psi$ satisfies property (**) then the components of the Néron model $X^{\mathrm{sm}}$ are in one-to-one correspondence with the set* $\{P \in Y \mid \text{the stabilizer of } P \text{ is } G\}$

*Proof:* For each subgroup $H \subseteq G$ we fix a resolution of singularities $A_H \to \mathbb{A}^d_{R'}/H$, and consider its formal completion $\widehat{A_H} \to \widehat{(\mathbb{A}^d_{R'}/H)}$.

Fix a point $P$, and consider the action of the stabilizer $G_P$ of $P$ on $Y$. Let $\tau_P$ be the translation by $P$ on $Y(k)$. Then $\tau_P$ induces an isomorphism $\tau_P^* : \overline{\mathbf{m}}_P/\overline{\mathbf{m}}_P^2 \to \overline{\mathbf{m}}_e/\overline{\mathbf{m}}_e^2$. Moreover, since $G$ is an abelian group, we have

$$\tau_P^* g^* (\tau_P^*)^{-1} = g^*$$

for all $g \in G_P$. It follows that the actions of $G_P$ on $\overline{\mathbf{m}}_P/\overline{\mathbf{m}}_P^2$ and $\overline{\mathbf{m}}_e/\overline{\mathbf{m}}_e^2$ are given by the *same* characters. We can thus use the formal resolution of singularities $\widehat{A_{G_P}} \to \widehat{(\mathbb{A}^d_{R'}/G_P)}$ to construct $Y/G_P$. If we use the same formal resolution at each point in the $G$-orbit of $P$, the action of $G/G_P$ identifies these formal resolutions. Thus, the quotient by $G/G_P$ gives a resolution of singularities of $Z$ (in a neighborhood of $\varphi(P)$), which has the same general form as the resolution of singularities $A_{G_P} \to \mathbb{A}^d_{R'}$. However, the multiplicities (in the fiber) of the components of the exceptional divisor change: they are multiplied by $|G/G_P|$. It follows that all components of multiplicity 1 come from points at which the stabilizer is the entire group $G$.

To check that $X$ is a good model of $A$, we define a differential form $\omega$ as follows. Choose a left-invariant differential form $\eta$ on $Y_{K'}$ which generates $\Omega^d_{Y/R'}$ at the generic point of the special fiber $Y_k$. In local parameters $y_1, \ldots, y_d$ near $Y_k$, $\eta$ takes the form $u(y_1, \ldots, y_d) dy_1 \wedge \cdots \wedge dy_d$, where $u(y_1, \ldots, y_d)$ is a unit in the local ring at that generic point.

Since $\chi_0$ is a primitive character of $G$ and $G$ is cyclic, there is some $k$ such that $\prod \chi_i = (\chi_0)^k$. Now $G$ acts on $\eta$ according to the character $\prod \chi_i$. Thus, $\widetilde{\eta} := z^{-k}\eta$ is a $G$-invariant differential. Let $\omega$ be the induced differential on $Z = Y/G$.

Since $A_G \to \mathbb{A}^d_{R'}/G$ satisfies property (*), there is some integer $n$ such that $\mathrm{ord}_{E_i}(\omega) = n$ for all components $E_i$ of $(A_G)_k$ with multiplicity 1. It follows that $\mathrm{ord}_{X_i}(\omega) = n$ for a smooth neighborhood $X_i$ of any of the components $E_i$ of $X_k$ with multiplicity 1. Thus, all such $X_i$ are $\Omega$-minimal.

In particular, we will have $\mathrm{ord}_{X_i}(\pi^{-n}\omega) = 0$ for all $X_i \subseteq X$ which are smooth $R$-models of $X_K$ with irreducible special fiber. Thus, $X$ is a good model. Q.E.D.

## 3 Resolution of certain quotient singularities

We continue to adhere to the notation introduced in the previous section. In particular, $R$ is a strictly henselian discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$. There is a totally ramified cyclic Galois extension $R'/R$ of degree $m$ prime to $p$ with Galois group $G$. We fix a uniformizing parameter $\pi$ for $R$ and identify $R'$ with $R[z]/(z^m - \pi)$. The character $\chi_0 : G \to \mu_m \subset R^\times$ defined by $\chi_0(g) = g(\sqrt[m]{\pi})/\sqrt[m]{\pi} \in \mu_m$ is independent of the choice of $\pi$.

**Theorem 3.1** *Let $Y = \mathrm{Spec}\, R'[x,y] = \mathrm{Spec}\, R[x,y][z]/(z^m - \pi)$, endowed with one of the following $G \cong \mathbb{Z}/m\mathbb{Z}$ actions, which we refer to as "unstarred type" and "starred type":*

$$g \in G \text{ acts by } (x,y,z) \mapsto \begin{cases} (\chi_0(g)x, \chi_0(g)y, \chi_0(g)z) & \text{for unstarred type} \\ (\chi_0(g)^{-1}x, \chi_0(g)^{-1}y, \chi_0(g)z) & \text{for starred type.} \end{cases}$$

*Let $\omega$ be the $G$-invariant differential on $Y_K$ defined as follows:*

$$\omega = \begin{cases} z^{-2} dx \wedge dy & \text{for unstarred type} \\ z^{2-2m} dx \wedge dy & \text{for starred type.} \end{cases}$$

*Let $Z = Y/G$, and denote the induced differential on $Z_K$ again by $\omega$.*

*Then there is a proper morphism $f : X \to Z$ of $R$-schemes with the following properties:*

1. *$X$ is regular,*

2. *the special fiber $X_k$ has exactly one component $E_1$ of multiplicity 1, and*

3. *the meromorphic differential $f^*(\omega)$ which $\omega$ induces on $X_K$ satisfies $\mathrm{ord}_{X_0}(\omega) = 0$, where $X_0$ is a smooth neighborhood of the generic point of $E_1$ in $X$.*

*Moreover, the set of regular points of $X_k$ (which all lie in $E_1$) is isomorphic to $\mathbb{A}^2_k$.*

*In particular, $f$ is a resolution of singularities which satisfies property* (**).

The special fiber $X_k$ is illustrated in Figure 1.

*Proof:* We construct $X$ by adapting the toroidal resolutions of cyclic quotient singularities (over $\mathbb{C}$) to the present case. For the cases at hand, the resolutions over $\mathbb{C}$ were worked out explicitly in [18], sect. 4 of part I, and [8].

For actions of unstarred type, we construct $X$ by means of three affine coordinate charts $X_i = \operatorname{Spec} R_i$, $i = 0, 1, 2$, as follows:

$$\begin{aligned} R_0 &= R[x_0, y_0, z_0]/(z_0 - \pi) \cong R[x_0, y_0] \\ R_1 &= R[x_1, y_1, z_1]/(x_1 z_1^m - \pi) \\ R_2 &= R[x_2, y_2, z_2]/(y_2 z_2^m - \pi) \end{aligned}$$

It is clear that each $\operatorname{Spec} R_i$ is regular, and that $X_0 = \operatorname{Spec} R_0$ is smooth over $R$.

We glue these affine charts together by means of the transition functions

$$\begin{aligned} x_0 &= z_1^{-1} &&= x_2 z_2^{-1} \\ y_0 &= y_1 z_1^{-1} &&= z_2^{-1} \\ z_0 &= x_1 z_1^m &&= y_2 z_2^m \end{aligned}$$

to obtain the scheme $X$. We then define a map $f : X \to Z$ by

$$f^*(x^\alpha y^\beta z^\gamma) = x_0^\alpha y_0^\beta z_0^{(\alpha+\beta+\gamma)/m} = x_1^{(\alpha+\beta+\gamma)/m} y_1^\beta z_1^\gamma = x_2^\alpha y_2^{(\alpha+\beta+\gamma)/m} z_2^\gamma. \tag{1}$$

To see that $f^*$ is well defined, note first that $x^\alpha y^\beta z^\gamma$ is $G$-invariant exactly when $\alpha + \beta + \gamma \equiv 0 \mod m$. Thus, $(\alpha + \beta + \gamma)/m$ is an integer. Note also that equation (1) implies that

$$f^*(z^m - \pi) = z_0 - \pi = x_1 z_1^m - \pi = y_2 z_2^m - \pi.$$

Thus, $f^*$ gives a well-defined map $(R[x, y, z]/(z^m - \pi))^G \to \bigcup R_i$.

We can describe the components of the special fiber $X_k$ as follows. There are two components:

$$\begin{aligned} E_1 &= \{z_0 = 0\} \cup \{x_1 = 0\} \cup \{y_2 = 0\}, \quad \text{and} \\ E_m &= \{z_1 = 0\} \cup \{z_2 = 0\}. \end{aligned}$$

It is clear from the defining equations for the charts that $E_1$ has multiplicity 1 and $E_m$ has multiplicity $m$. Moreover, $E_1 \setminus E_m \subset X_0$ and $E_m \cap X_0 = \emptyset$, as is easily seen from the transition functions. Thus, $E_1 \setminus E_m = (X_0)_k = \operatorname{Spec} k[x_0, y_0] \cong \mathbb{A}^2_k$ and this is exactly the set of regular points of $X$, that is, $X^{\mathrm{sm}} = X_0$. ($E_1$ itself is isomorphic to $\mathbb{P}^2_k$.)

$X_0$ is a smooth $R$-model with irreducible special fiber, and we must compute $\mathrm{ord}_{X_0}(\omega)$. If we localize at the generic point of $E_1$, then $z_0$ serves as a uniformizing parameter for the corresponding discrete valuation ring. Now we compute $f^*(\omega)$ as follows. Since $z \in K$, $dz = 0$. By equation (1), $f^*(z^{-1}x) = x_0$ and $f^*(z^{-1}y) = y_0$. Therefore, $f^*(z^{-1}dx) = dx_0$, $f^*(z^{-1}dy) = dy_0$, which implies that $f^*(\omega)|_{X_0} = dx_0 \wedge dy_0$. It is then clear that $f^*(\omega)$ is itself a generator of $\Omega^2_{X_0/R}$, so that $\mathrm{ord}_{X_0}(f^*(\omega)) = 0$.

Finally, we must show that $f : X \to Z$ is proper. To do this, we consider an arbitrary discrete valuation ring $S$ together with an arbitrary map $\mathrm{Spec}\, S \to Z$. Such a map determines orders of vanishing of the coordinate functions, namely, $a := v_S(x)$, $b := v_S(y)$, $c := v_S(z)$. The corresponding rational map $\mathrm{Spec}\, S \to X$ then has the following valuative properties:

$$\begin{array}{lll} v_S(x_0) = a - c, & v_S(y_0) = b - c, & v_S(z_0) = mc; \\ v_S(x_1) = ma, & v_S(y_1) = b - a, & v_S(z_1) = c - a; \\ v_S(x_2) = a - b, & v_S(y_2) = mb, & v_S(z_2) = c - b. \end{array}$$

Thus, for $a, b, c \geq 0$, if $\min\{a, b, c\} = a$ (resp. $b$, resp. $c$) then the map extends to $\mathrm{Spec}\, S \to X_1$ (resp. $\mathrm{Spec}\, S \to X_2$, resp. $\mathrm{Spec}\, S \to X_0$). Hence $f$ is proper by the valuative criterion.

We now turn to actions of starred type. In this case, we construct $X$ by means of $2m-1$ affine coordinate charts $X_i = \mathrm{Spec}\, R_i$, $1 - m \leq i \leq m - 1$, as follows:

$$\begin{array}{rcl} R_0 & = & R[x_0, y_0, z_0]/(z_0 - \pi) \cong R[x_0, y_0] \\ R_i & = & R[r_i, s_i, t_i]/(r_i^i s_i^{i+1} - \pi) \ \text{ for } 1 \leq i \leq m-1 \\ R_{-i} & = & R[u_i, v_i, w_i]/(u_i^i v_i^{i+1} - \pi) \ \text{ for } 1 \leq i \leq m-1. \end{array}$$

It is clear that each $\mathrm{Spec}\, R_i$ is regular, and that $X_0 = \mathrm{Spec}\, R_0$ is smooth over $R$.

We glue these affine charts together by means of the transition functions

$$\begin{array}{rcccl} x_0 & = & r_i^{1-i} s_i^{-i} & = & u_i^{1-i} v_i^{-i} w_i \\ y_0 & = & r_i^{1-i} s_i^{-i} t_i & = & u_i^{1-i} v_i^{-i} \\ z_0 & = & r_i^i s_i^{i+1} & = & u_i^i v_i^{i+1} \end{array}$$

to obtain the scheme $X$. We then define a map $f : X \to Z$ by

$$(2) \qquad \begin{array}{rcl} f^*(x^\alpha y^\beta z^\gamma) & = & x_0^\alpha y_0^\beta z_0^{((m-1)(\alpha+\beta)+\gamma)/m} \\ & = & r_i^{((m-i)(\alpha+\beta)+i\gamma)/m} s_i^{((m-i-1)(\alpha+\beta)+(i+1)\gamma)/m} t_i^\beta \\ & = & u_i^{((m-i)(\alpha+\beta)+i\gamma)/m} v_i^{((m-i-1)(\alpha+\beta)+(i+1)\gamma)/m} w_i^\alpha. \end{array}$$

To see that $f^*$ is well defined, note first that $x^\alpha y^\beta z^\gamma$ is $G$-invariant exactly when $\alpha + \beta - \gamma \equiv 0 \mod m$. Thus, $((m-i)(\alpha+\beta)+i\gamma)/m$ is an integer. Note also that equation (2) implies that

$$f^*(z^m - \pi) = z_0 - \pi = r_i^i s_i^{i+1} - \pi = u_i^i v_i^{i+1} - \pi.$$

Thus, $f^*$ gives a well-defined map $(R[x,y,z]/(z^m-\pi))^G \to \bigcup R_i$.

We can describe the components of the special fiber $X_k$ as follows. There are $m$ components:

$$\begin{aligned} E_1 &= \{z_0=0\} \cup \{r_1=0\} \cup \{u_1=0\}, \\ E_i &= \{r_i=0\} \cup \{s_{i-1}=0\} \cup \{u_i=0\} \cup \{v_{i-1}=0\}, \quad 2 \le i \le m-1, \\ E_m &= \{s_{m-1}=0\} \cup \{v_{m-1}=0\}. \end{aligned}$$

It is clear from the defining equations for the charts that $E_i$ has multiplicity $i$. Moreover, $E_1 \setminus E_2 \subset X_0$ and $E_i \cap X_0 = \emptyset$ for all $i > 1$, as is easily seen from the transition functions. Thus, $E_1 \setminus E_2 = (X_0)_k = \operatorname{Spec} k[x_0, y_0] \cong \mathbb{A}^2_k$ and this is exactly the set of regular points of $X$, that is, $X^{\mathrm{sm}} = X_0$. ($E_1$ itself is isomorphic to $\mathbb{P}^2_k$.)

$X_0$ is a smooth $R$-model with irreducible special fiber, and we must compute $\operatorname{ord}_{X_0}(\omega)$. If we localize at the generic point of $E_1$, then $z_0$ serves as a uniformizing parameter for the corresponding discrete valuation ring. Now we compute $f^*(\omega)$ as follows. Since $z \in K$, $dz = 0$. By equation (2), $f^*(z^{1-m}x) = x_0$ and $f^*(z^{1-m}y) = y_0$. Therefore, $f^*(z^{1-m}dx) = dx_0$, $f^*(z^{1-m}dy) = dy_0$, which implies that $f^*(\omega)|_{X_0} = dx_0 \wedge dy_0$. It is then clear that $f^*(\omega)$ is itself a generator of $\Omega^2_{X_0/R}$, so that $\operatorname{ord}_{X_0}(f^*(\omega)) = 0$.

Finally, we must show that $f : X \to Z$ is proper. To do this, we consider an arbitrary discrete valuation ring $S$ together with an arbitrary map $\operatorname{Spec} S \to Z$. Such a map determines orders of vanishing of the coordinate functions, namely, $a := v_S(x)$, $b := v_S(y)$, $c := v_S(z)$. The corresponding rational map $\operatorname{Spec} S \to X$ then has the following valuative properties:

$$\begin{array}{lll} v_S(x_0) = a-(m-1)c, & v_S(y_0) = b-(m-1)c, & v_S(z_0) = mc; \\ v_S(r_i) = (i+1)a-(m-i-1)c, & v_S(s_i) = -ia+(m-i)c, & v_S(t_i) = b-a; \\ v_S(u_i) = (i+1)b-(m-i-1)c, & v_S(v_i) = -ib+(m-i)c, & v_S(w_i) = a-b. \end{array}$$

Thus, for $a,b,c \ge 0$, if $\min\{a,b,(m-1)c\} = (m-1)c$ then the map extends to $\operatorname{Spec} S \to X_0$, while if $\min\{a,b,(m-1)c\} = a$ (resp. $b$) and

$$\frac{m}{i+1} \le 1 + \frac{a}{c} \le \frac{m}{i} \quad \left(\text{resp. } \frac{m}{i+1} \le 1 + \frac{b}{c} \le \frac{m}{i}\right)$$

then the map extends to $\operatorname{Spec} S \to X_i$ (resp. $\operatorname{Spec} S \to X_{i+m-1}$). (Note that this covers all cases, since $1 + (a/c) \le 1 + (m-1) = m$.)

Hence $f$ is proper by the valuative criterion. Q.E.D.

We can now deduce the analogous result for surface singularities (which should be well known) by using the constructions in the previous proof.

**Theorem 3.2** *Let $T = \operatorname{Spec} R'[y] = \operatorname{Spec} R[y][z]/(z^m - \pi)$, endowed with one of the following $G \cong \mathbb{Z}/m\mathbb{Z}$ actions, which we refer to as "unstarred type" and "starred type":*

$$g \in G \text{ acts by } (y, z) \mapsto \begin{cases} (\chi_0(g)y, \chi_0(g)z) & \text{for unstarred type} \\ (\chi_0(g)^{-1}y, \chi_0(g)z) & \text{for starred type.} \end{cases}$$

*Let $\alpha$ be the $G$-invariant differential on $T_K$ defined as follows:*

$$\alpha = \begin{cases} z^{-1}dy & \text{for unstarred type} \\ z^{1-m}dy & \text{for starred type.} \end{cases}$$

*Let $U = T/G$, and denote the induced differential on $U_K$ again by $\alpha$.*

*Then there is a proper morphism $\varphi : S \to U$ of $R$-schemes with the following properties:*

1. *$S$ is regular,*

2. *the special fiber $S_k$ has exactly one component $D_1$ of multiplicity $1$, and*

3. *the meromorphic differential $\varphi^*(\alpha)$ which $\alpha$ induces on $S_K$ satisfies $\operatorname{ord}_{S_0}(\alpha) = 0$, where $S_0$ is a smooth neighborhood of the generic point of $D_1$ in $X$.*

*Moreover, the set of regular points of $S_k$ (which all lie in $D_1$) is isomorphic to $\mathbb{A}^1_k$.*

*In particular, $\varphi$ is a resolution of singularities which satisfies property* (**).

*Proof:* We use the constructions from the proof of Theorem 3.1, retaining the notation from that proof. We can embed $T$ in $Y$ as the hypersurface defined by $x = 0$. We then construct $S \subset X$ by defining

$$S = \begin{cases} \{z_0 = 0\} \cup \{x_1 = 0\} \cup \{y_2 = 0\} & \text{for unstarred type} \\ \{x_0 = 0\} \cup \{w_i = 0\} & \text{for starred type.} \end{cases}$$

($S$ does not meet the coordinate charts $R_i$ for $1 \leq i \leq m-1$.) $S$ is clearly regular, with $f(S) = T \subset Y$. The components of the special fiber of $S$ are $D_i = E_i \cap S$, and the smooth neighborhood $S_0$ of $D_1$ is provided by $S_0 = X_0 \cap S$. These things satisfy the conditions in the theorem, just as in the previous proof. Moreover, the previous proof of properness applies to this case as well.

The only thing to check is the order of the form $\alpha$ at $S_0$. We calculate

$$\varphi^*(\alpha) = \begin{cases} \varphi^*(z^{-1}dy) = dy_0 & \text{for unstarred type} \\ \varphi^*(z^{1-m}dy) = dy_0 & \text{for starred type.} \end{cases}$$

Thus, $\varphi^*(\alpha)$ is a generator of $\Omega^2_{S_0/R}$, so that $\operatorname{ord}_{X_0}(\varphi^*(\alpha)) = 0$. Q.E.D.

The illustration in Figure 1 includes the surface $S \subseteq X$, and the components $D_i$ of the special fiber $S_k$ of $S$.

## 4 Abelian varieties with prescribed endomorphisms

Let $R$ be a henselian discrete valuation ring with fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of characteristic $p$. Let $K_s$ be a separable closure of $K$. Suppose $A/K$ is an abelian variety with potential good reduction and endomorphism structure $\mathcal{M} \hookrightarrow \mathrm{End}_K(A)$ for a ring $\mathcal{M}$. Let $K'/K$ be the extension cut out by the representation

$$\rho : \mathrm{Gal}(K_s/K) \longrightarrow \mathrm{Aut}_{\mathcal{M}}(\mathrm{Ta}_\ell(A)), \quad \ell \neq p.$$

Then $\mathrm{Gal}(K'/K)$ is isomorphic to $\Phi(A/K)$, and $A \times_K K'/K'$ has good reduction. Denote by $R'$ the integral closure of $R$ in $K'$ and by $Y/R'$ a Néron model of $A \times_K K'$ over $R'$ with special fiber $Y_0/k = Y \times_{R'} k$. Let $\mathcal{F} = \mathcal{F}(A/K) \subseteq Y_0(k)$ be the set of points fixed by $G$. Observe that $\mathcal{F}$ is an $\mathcal{M}$-submodule of $Y_0(k)$.

We shall study in detail endomorphism structures of the following type:

(HYP) $\mathrm{End}^0_{\mathcal{M}}(Y_0/k) = \mathrm{End}_{\mathcal{M}}(Y_0/k) \otimes \mathbb{Q}$ is either an imaginary quadratic field or a rational definite quaternion algebra.

In this case we bound $\Phi(A/K)$ using the following:

**Lemma 4.1** *Suppose $H$ is a finite subgroup of the multiplicative group of either an imaginary quadratic field or a rational definite quaternion algebra. Then $|H|$ divides 24. If furthermore $H$ is abelian, then $|H| = 1$, 2, 3, 4, or 6.*

*Proof:* By [19], Proposition 3.1, p. 145, if such a group $H$ is not cyclic of order 1, 2, 3, 4, or 6, then it is a subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ or of the binary dihedral group ("groupe dicyclique") of order 12. As $|\mathrm{SL}_2(\mathbb{F}_3)| = 24$, this shows that $|H|$ divides 24. Any abelian subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ or of the binary dihedral group of order 12 must be cyclic of order 1, 2, 3, 4, or 6. This gives the desired classification if $H$ is abelian, and concludes the proof. Q.E.D.

**Proposition 4.2** *Suppose the abelian variety $A/K$ with potential good reduction satisfies* (HYP) *and $p > 3$. Then $\Phi(A/K)$ is cyclic of order* 1, 2, 3, 4, *or* 6.

*Proof:* The group $\Phi(A/K) \subseteq \mathrm{Aut}_{\mathcal{M}}(Y_0/k)$ is a finite subgroup of $\mathrm{End}^0_{\mathcal{M}}(Y_0/k)$. Hence, $|\Phi(A/K)|$ divides 24 by Lemma 4.1. If $p > 3$ then the totally ramified extension $K'/K$ is tamely ramified and hence abelian. By Lemma 4.1, $\Phi(A/K)$ is cyclic of order 1, 2, 3, 4, or 6. Q.E.D.

Two examples of (HYP) are:

(1) $A/K$ is an elliptic curve with potential good reduction and $\mathcal{M} = \mathbb{Z}$.

(2) $\mathcal{M}$ is an order in an indefinite rational quaternion division algebra and $A/K$ is an abelian surface admitting $i : \mathcal{M} \hookrightarrow \mathrm{End}_K(A)$. In this case we say that $(A, i)/K$ is a QM-abelian surface. Such abelian surfaces always have potential good reduction, cf. [4], Exposé III. See [12], Proposition 2.3, for the proof that a QM-abelian surface $(A, i)/K$ satisfies (HYP).

In fact, the only simple abelian varieties satisfying (HYP) are of type (1) or (2).

**Theorem 4.3** *Suppose $A/K$ is an abelian variety of dimension $d$ with potential good reduction satisfying* (HYP) *and $p > 3$. Suppose $G \cong \Phi(A/K) \cong \mathbb{Z}/m\mathbb{Z}$ with $m > 1$. For each $\mu > 1$ which divides $m$, let $G_\mu \subseteq G$ be the subgroup of order $\mu$. Let $Y_0[n] \subseteq Y_0(k)$ denote the set of $n$-torsion points on the abelian variety $Y_0$ for $(n, p) = 1$.*

1. *If $\mathcal{F}_\mu$ is the set of points fixed by $G_\mu$, then $\mathcal{F}_\mu \subseteq Y_0[n_\mu]$, where $n_2 = 2$, $n_3 = 3$, $n_4 = 2$, and $n_6 = 1$. Moreover, the cardinality of $\mathcal{F}_\mu$ is $f_\mu$, where $f_2 = 4^d$, $f_3 = 3^d$, $f_4 = 2^d$, and $f_6 = 1$.*

2. *If $N_\mu$ is the number of $(G/G_\mu)$-orbits of points whose stabilizer is $G_\mu$, then $N_\mu$ is determined by $m = \#(\Phi(A/K))$ as shown in the second column of Table 1.*

*Proof:* If $\mu = 2$, then $G_\mu = \{\pm 1\}$. The fixed points are all 2-torsion points, and there are $4^d$ of these; hence $f_2 = 4^d$ and $n_2 = 2$.

If $G_\mu = \mathbb{Z}/3\mathbb{Z}$, then $\mathbb{Q}[\rho(G_\mu)] \cong \mathbb{Q}(\sqrt{-3}) \subseteq \mathrm{End}^0_{\mathcal{M}}(Y_0/k)$. For a generator $g$ of $G_\mu$, $\rho(g) = \omega \in \mathbb{Q}(\sqrt{-3})$ is a cube root of unity. The fixed points $\mathcal{F}$ of $G_\mu$ acting on $Y_0(k)$ are then the kernel of $(\omega - 1) \in \mathrm{End}_{\mathcal{M}}(Y_0/k)$. The cardinality $|\mathcal{F}_\mu|$ of $\mathcal{F}_\mu$ is then the constant term $a_0$ of the (degree $2d$) characteristic polynomial of $(\omega - 1)$ acting on $\mathrm{Ta}_\ell(Y_0)$. But $a_0 = \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(\omega - 1)^d = 3^d$. Visibly, the kernel of $(\omega - 1)$ is contained in $Y_0[\mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(\omega - 1)] = Y_0[3]$.

The proof in the case $\mu = 4$ is similar: if $G_\mu = <g> \cong \mathbb{Z}/4\mathbb{Z}$, then $\rho(g) = i \in \mathrm{End}^0_{\mathcal{M}}(Y_0/k)$ is a fourth root of unity. And $\mathrm{N}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}(i - 1) = 2$ implying that $|\mathcal{F}_\mu| = 2^d$ and $\mathcal{F}_\mu \subseteq Y_0[2]$.

As for the case $\mu = 6$, if $G_\mu = <g> \cong \mathbb{Z}/6\mathbb{Z}$, then $\rho(g) = -\omega \in \mathbb{Q}(\sqrt{-3})$ is a sixth root of unity. But $\mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-\omega - 1) = 1$, showing that $|\mathcal{F}_\mu| = 1$.

To prove (2), note that the fixed points of $G_\mu$ come in two varieties: those fixed by all of $G$, and those which belong to a nontrivial $(G/G_\mu)$-orbit. Since each such orbit has $m/\mu$ points ($m/\mu$ being 1 or a prime in all of our cases), we see that

$$\left(\frac{m}{\mu}\right) N_\mu + f_m = f_\mu.$$

The entries in the table follow. Q.E.D.

## 5 Néron models of elliptic curves and of abelian surfaces with quaternionic multiplication

We now describe how the theory we have developed can be used to recover the Kodaira–Néron classification of elliptic curves with potentially good reduction, and to give a similar classification of Néron models for abelian surfaces with quaternionic multiplication. We work over a henselian discrete valuation ring $R$ with an algebraically closed residue field $k$ of characteristic $p$. Denote by $K$ the field of fractions of $R$ and by $K_s$ a separable closure of $K$. Let $A$ be either an elliptic curve over $R$ with potentially good reduction, or an abelian surface over $R$ with quaternionic multiplication. Let $d$ be the dimension of $A$. Let $K'/K$ be the Galois extension cut out by the representation

$$\mathrm{Gal}(K_s/K) \longrightarrow \mathrm{Aut}(\mathrm{Ta}_\ell(A)).$$

Then $G = \mathrm{Gal}(K'/K)$ is isomorphic to the image $\Phi(A/K)$ of $\mathrm{Gal}(K_s/K)$ in the automorphism group $\mathrm{Aut}(\mathrm{Ta}_\ell(A))$. Denote by $R'$ the integral closure of $R$ in $K'$ and by $Y/R'$ a Néron model of $A$ over $R'$. We assume that $p \neq 2$ or 3.

As we observed in section 4, our abelian variety satisfies (HYP). It then follows from Lemma 4.1 that the group $G \cong \Phi(A/K)$ is cyclic of order $m \in \{1, 2, 3, 4, 6\}$.

If $m = 1$, then the curve $A$ has good reduction. Otherwise, the origin $e \in Y(k)$ is an isolated fixed point for the action of $G$. There are therefore characters $\chi_1, \dots, \chi_d$ of $G$ determined by the action of $G$ on the tangent space to $e$, as in Lemma 2.4. In fact, in our cases all of these characters are equal, since $\mathrm{Ta}_\ell(A) \otimes \mathbb{Q}_\ell$ is a rank one $(\mathcal{M} \otimes \mathbb{Q}_\ell)$-module.

In order to apply the construction from Theorem 2.6, we need a resolution of the quotient singularities whose determining characters are $(\chi_0, \chi_1)$ or $(\chi_0, \chi_1, \chi_1)$, where $\chi_0$ is the canonical character for the action of $G$ on $R$. But since $G$ is a cyclic group of order $m \in \{2, 3, 4, 6\}$, we have $\chi_1 = \chi_0^{\pm 1}$, so the resolutions of singularities from Theorems 3.1 and 3.2 can be used. We say that the abelian variety $A$ is of *starred type* if $\chi_1 = \chi_0^{-1}$, and *unstarred type* if $m > 2$ and $\chi_1 = \chi_0$. (This is consistent with the usage in Theorems 3.1 and 3.2.)

We calculated the fixed point structure for the group action in Theorem 4.3, displaying the result in Table 1. The third and fourth columns of that table establish notation for the various possible cases, consistent with Kodaira's notation for elliptic curves. In particular, a "star" is used to denoted starred cases.

Theorem 2.6 can now be used to produce a good model $X$ and a Néron model $X^{\text{Nér}}$, when the abelian variety is of starred type (resp. unstarred type). The special fiber $X_k$ of the good model can be described as follows. There is a component $F$ of multiplicity $m$ which is the image of the special fiber $Y_k$ of $Y$. In addition, for each $(G/G_\mu)$-orbit of points with stabilizer $G_\mu$, there are the components of $X_k$ which correspond to a resolution of a quotient singularity of order $\mu$, of starred type (resp.

unstarred type). These resolutions are as described in Theorem 3.1 in dimension $d = 2$, and in Theorem 3.2 in dimension $d = 1$, except that the multiplicities of the components appearing in the resolution must be multiplied by $m/\mu$. In particular, all components of multplicity 1 in $X_k$ come from points with stabilizer $G_m = G$, and there is one such component for each point. These are exactly the components of the central fiber of the Néron model, so the group of connected components is isomorphic to the subgroup $\mathcal{F}_m$ of the special fiber (indicated as $\Gamma^{\text{Nér}}(A/K)$ in the last column of Table 1).

The description in the previous paragraph can perhaps best be visualized if figures are drawn illustrating the special fibers. For elliptic curves, these figures are very familiar, and can be found in Tate's paper [17] among other places. For abelian surfaces, all of the relevant illustrations appear in Ueno's work [18]. We have included sample illustrations as Figures 2 and 3, sketching the special fibers for $d = 1$ and $d = 2$ in all cases with $m = 4$. The reader should be able to create similar illustrations based on Table 1 for the other cases.

The third and fourth columns of Table 1 establish notation for the various possible cases, consistent with Kodaira's notation for elliptic curves. In particular, a "star" is used to denoted starred cases.

For elliptic curves of starred type, the proper regular model constructed here is the minimal model of the elliptic curve. However, for elliptic curves of unstarred stype, our model is not the minimal model but rather the smallest birational model for which the special fiber is a divisor with normal crossings. The minimal model itself can be recovered by contracting to a point all components of the special fiber of multiplicity greater than 1. The result of this operation is shown in Figure 4. Note that all components of the Néron model still appear in the minimal model.

In the case $d = 2$, it is instructive to make a similar comparison between the models constructed here and the Mori minimal models for the analogous complex analytic degenerations (as worked out in [5]). The construction is similar to the elliptic curve case: starting from a resolution of singularities $X \to Y/G$, certain birational operations (including the contractions of some components of the special fiber to points) must be performed in order to produce the Mori minimal model from $X$. However, these operations do not always respect the Néron model, some of whose components may be contracted to points. Thus, the smooth regular locus of the Mori minimal model may fail to entirely contain the Néron model.

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| $\#(\Phi(A/K))$ | $(N_2, N_3, N_4, N_6)$ | starred type | unstarred type | $\Gamma^{\text{Nér}}(A/K)$ |
|---|---|---|---|---|
| 2 | $(4^d, 0, 0, 0)$ | $I_0^*$ | — | $(\mathbb{Z}/2\mathbb{Z})^{2d}$ |
| 3 | $(0, 3^d, 0, 0)$ | $IV^*$ | $IV$ | $(\mathbb{Z}/3\mathbb{Z})^d$ |
| 4 | $(\frac{4^d-2^d}{2}, 0, 2^d, 0)$ | $III^*$ | $III$ | $(\mathbb{Z}/2\mathbb{Z})^d$ |
| 6 | $(\frac{4^d-1}{3}, \frac{3^d-1}{2}, 0, 1)$ | $II^*$ | $II$ | trivial |

Table 1: Fixed points and the group of components

**Starred type:**

$E_m$

$E_{m-1}$ $\cdots$ $E_1$

$D_{m-1}$ $D_1$

$S$

$D_m$

**Unstarred type:**

$E_m$

$E_1$

$D_1$

$S$

$D_m$

Figure 1: Resolutions of quotient singularities

$\mu = 2$

$\mu = 4$

$\mu = 4$

Type III* Type III

Figure 2: Elliptic curves with $m = 4$

$\}\mu = 4$

$\}\mu = 2$

$F$ $F$

Type III* Type III

Figure 3: QM-abelian surfaces with $m = 4$

| Type | Our model | minimal model |
|---|---|---|
| | $\mu = 3$ | |
| IV | $\mu = 3$ | |
| | $\mu = 3$ | |
| | $\mu = 2$ | |
| III | $\mu = 4$ | |
| | $\mu = 4$ | |
| | $\mu = 2$ | |
| II | $\mu = 3$ | |
| | $\mu = 6$ | |

Figure 4: Minimal models