

LINEAR SYSTEMS ON  $K3$ -SECTIONS

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## 1. Introduction

The types of special linear systems which exist on a curve  $C$  which is a hyperplane section of a  $K3$  surface  $X$  often do not depend on  $C$  but only on its linear equivalence class in  $X$ . For instance, Saint-Donat proved in [14] that  $C$  possesses a  $g_2^1$  or  $g_3^1$  if and only if the same is true for every nonsingular curve  $C' \in |C|$ , where  $|C|$  denotes the linear system of  $C$  on  $X$ , and Reid [12] found some extensions of this result to other  $g_d^1$ 's. The general question of whether the presence of a special  $g_d^r$  on a given hyperplane section  $C$  of a  $K3$  surface forces the existence of such a  $g_d^r$  on every nonsingular  $C' \in |C|$  arose out of work of Harris and Mumford [7]. Our purpose is to study this question and some related conjectures. We use the term  *$K3$ -section* to denote a smooth curve of genus at least two on a  $K3$  surface. (Such a curve, if nonhyperelliptic, is a hyperplane section of a birational model of the  $K3$  surface  $X$  in some projective embedding.)

We start, in §2, with a counterexample: a  $K3$  surface  $X$  in  $\mathbf{P}^{10}$ , some of whose hyperplane sections (but not all) possess a  $g_4^1$ . In §3 we use a counting argument to show that if  $C$  carries a  $g_d^1$  which is scheme-theoretically isolated in moduli, then this  $g_d^1$  "propagates" to every nonsingular  $C' \in |C|$ , in the sense that an explicit geometric construction starting from the  $g_d^1$  on  $C$  produces a  $g_d^1$  on  $C'$ . A sufficient condition for the propagation of  $g_d^r$ 's is also obtained, but it is weak for  $r > 1$ .

Analysis of our counterexample shows that in the family of all nonsingular hyperplane sections of  $X$ , the subfamily of curves carrying a  $g_4^1$  has codimension one. On the other hand, all these curves *do* carry a  $g_6^2$ . Combining this observation with his theory of Koszul cohomology, Mark Green suggested that the correct conjecture is not propagation of  $g_d^r$ 's but constancy of the "Clifford index"  $\nu = d - 2r$ . More precisely, for a line bundle  $M$  on a  $K3$ -section  $C$  with  $h^0(M) = r + 1$ ,  $\deg(M) = d$ , and  $\text{genus}(C) = g$ , define

$$\nu(M) := d - 2r, \quad \nu(C) := \min\{\nu(M) \mid r \geq 1, d \leq g - 1\}.$$

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Clifford's theorem says that  $\nu(C) \geq 0$ , with equality if and only if  $C$  is hyperelliptic. We also define

$$\nu(\mathcal{O}_X(C)) := \nu(C') \quad \text{for generic } C' \in |C|.$$

(Notice that the function  $C' \mapsto \nu(C')$  is lower semicontinuous on the family of nonsingular curves  $C' \in |C|$ , so that  $\nu(\mathcal{O}_X(C))$  can be characterized as the smallest integer  $\nu$  such that for every nonsingular  $C' \in |C|$  there is some line bundle  $M'$  on  $C'$  with  $h^0(M') \geq 2$ ,  $\deg(M') \leq g-1$  and  $\nu(M') \leq \nu$ .) Green's conjecture is then:

**(1.1) Conjecture [3].** *If  $X$  is a K3 surface and  $L$  is an ample line bundle on  $X$  then  $\nu(C) = \nu(L)$  for all nonsingular  $C \in |L|$ .*

In §4 we prove this conjecture for  $g_d^1$ 's. That is, we show that if the Clifford index of a nonsingular  $C$  is achieved by a  $g_d^1$ , i.e., if there is a  $g_d^1$  on  $C$  with  $d-2 = \nu(C)$ , then  $\nu(C) = \nu(\mathcal{O}_X(C))$ . Reid [12] had earlier shown this when  $g$  is sufficiently large with respect to  $d$ .

Another interesting feature of our counterexample is that the  $g_6^2$  linear systems on all the hyperplane sections  $C' \in |C|$  are restrictions of one and the same line bundle on  $X$ ; the same holds for the  $g_2^1$ 's and  $g_3^1$ 's studied by Saint-Donat. In a second counterexample, based on an example of Reid [12], we exhibit a K3 surface  $X$  with an ample linear system  $|C|$  such that every  $C' \in |C|$  has a  $g_6^1$ , but these are not all induced from the same bundle on  $X$ . (For generic  $C' \in |C|$ , these  $g_6^1$ 's are scheme-theoretically isolated in moduli and have negative Brill-Noether number  $\rho < 0$ , but are not unique.) Again, each of these  $g_6^1$ 's is contained in a  $g_8^2$  (which the reader should notice has the same Clifford index  $\nu = 4$ ), and these  $g_8^2$ 's are induced from a bundle on  $X$ . We suggest that this is a general phenomenon:

**(1.2) Conjecture.** *Let  $X$  be a K3 surface,  $C$  be a smooth curve on  $X$  of genus  $g \geq 2$ , and  $|Z|$  be a complete base point free  $g_d^r$  on  $C$  with  $r \geq 1$ ,  $d \leq g-1$ , such that*

$$\rho(Z) := (d-r)(r+1) - rg < 0.$$

*Then the linear system  $|Z|$  is contained in the restriction to  $C$  of a linear system  $|D|$  on  $X$  with*

$$\deg(D \cap C) \leq g-1, \quad \nu(D \cap C) \leq \nu(Z).$$

(We recall that a linear system  $|Z|$  on  $C$  is *contained in* another system  $|Z'|$  if every divisor  $Z \in |Z|$  is contained in some  $Z' \in |Z'|$ , i.e.,  $Z \leq Z'$  as divisors on  $C$ .)

Conjecture (1.2) clearly implies (1.1); this requires an easy computation which we leave to the reader. In §5 we extend the analysis of §4, proving (1.2)

for  $r = 1$ . Once again, the first results in this direction are due to Reid [12], who used Ramanujam's theory of numerical connectedness of divisors on a surface [11]. Our technique in §§4 and 5 is somewhat different: inspired by work of Lazarsfeld [8] and Reider [13], we construct a rank two vector bundle on  $X$  in order to study the  $g_d^1|Z|$ .

After this work had been completed (but before this paper was finished), we received a preprint from Green and Lazarsfeld [4], which proves Green's conjecture (1.1) in full generality, and also a part of (1.2): there is a linear system  $|D|$  on  $X$  such that  $\nu(\mathcal{O}_C(D)) = \nu(C)$ . From that preprint we also learned of some work of Tyurin [15] related to our construction in §3.

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**2. Linear systems on  $K3$  surfaces: review and counterexamples**

(2.1) We gather here some useful facts about linear systems on a  $K3$  surface  $X$ , taken from Mayer [9] and Saint-Donat [14]. To start, we list some examples of exceptional behavior:

X1. Let  $F \subset X$  be a smooth elliptic curve, and consider  $L := \mathcal{O}(kF)$ ,  $k \geq 1$ . We then have

$$h^0(L) = k + 1, \quad h^1(L) = k - 1,$$

and the map  $\varphi_{|L|}$  determined by sections of  $L$  sends  $X$  to a rational normal curve in  $\mathbf{P}^k$ . In particular, all divisors in  $|L|$  are of the form  $\sum_{i=1}^k F_i$  with  $F_i \sim F$ .

X2. Let  $\Gamma \subset X$  be a smooth rational curve,  $F \subset X$  smooth elliptic as above, and  $\Gamma \cdot F = 1$ . Consider  $L := \mathcal{O}(kF + \Gamma)$ ,  $k \geq 2$ . We then have

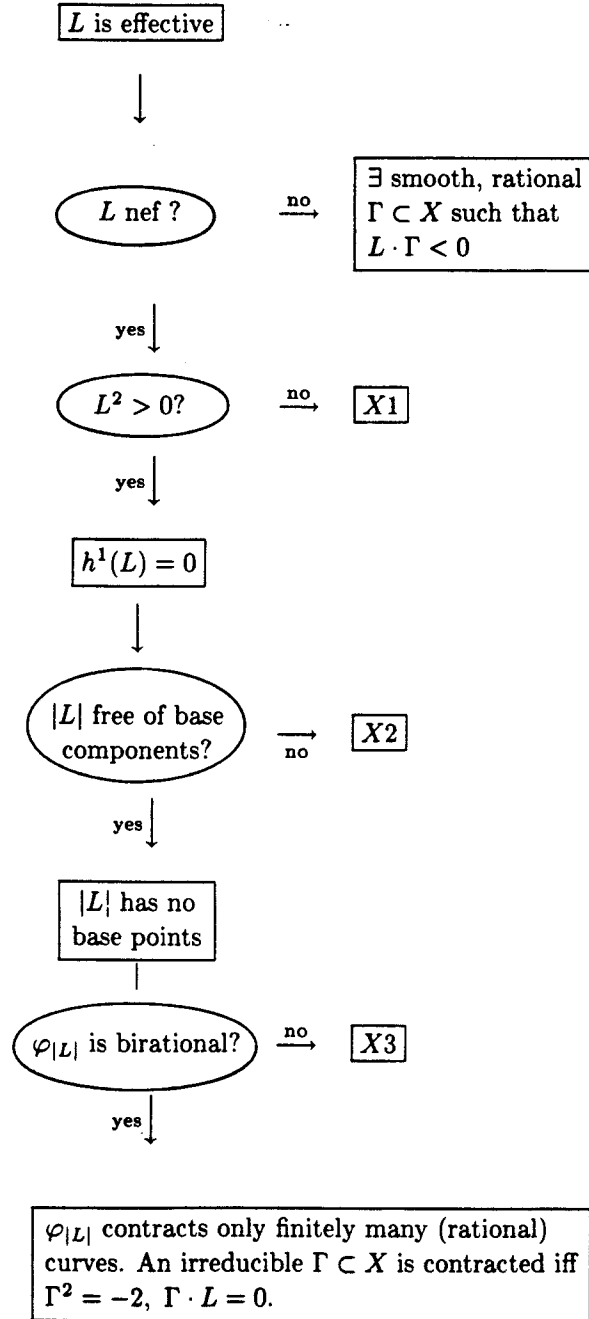
$$h^0(L) = k + 1, \quad h^1(L) = 0,$$

and all divisors in  $|L|$  are of the form  $\Gamma + \sum_{i=1}^k F_i$  with  $F_i \sim F$ , so  $\varphi_{|L|}$  has base-component  $\Gamma$  and maps  $X$  to a rational normal curve in  $\mathbf{P}^k$ .

X3. Let  $D \subset X$  be a smooth hyperelliptic curve of genus  $g \geq 2$ , and let  $L := \mathcal{O}(D)$ . Then  $\varphi_{|L|}$  is two-to-one, and every divisor in  $|L|$  is hyperelliptic. If  $(n - 1)(g - 1) > 1$ , then the map  $\varphi_{|nL|}$  is birational.

In a sense, these are the only cases of exceptional behavior. More precisely, let  $L$  be an effective line bundle on  $X$ . The properties of  $|L|$  can be read off

the following flow chart:



(2.2) **A counterexample: nonpropagating  $g_4^1$ 's.** Let  $\pi : X \rightarrow \mathbf{P}^2$  be a K3 surface of genus 2, i.e. a double cover of  $\mathbf{P}^2$  branched along a nonsingular plane sextic curve  $B \subset \mathbf{P}^2$ . The line bundle of degree 2 given by  $\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$  is then just a special case of example X3. Instead we take  $L := \pi^*\mathcal{O}_{\mathbf{P}^2}(3)$ . We claim:

- (i)  $\varphi_{|L|}: X \rightarrow \mathbf{P}^{10}$  is an embedding.
- (ii) There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{|L|}} & \mathbf{P}^{10} \\ \pi \downarrow & & \downarrow \text{pr} \\ \mathbf{P}^2 & \xrightarrow{v} & \mathbf{P}^9 \end{array}$$

where  $v$  is the Veronese embedding of  $\mathbf{P}^2$  in  $\mathbf{P}^9$  via the complete linear system  $|\mathcal{O}_{\mathbf{P}^2}(3)|$ , and  $\text{pr}$  is a linear projection.

(iii) Any hyperplane section of  $X$  which comes from  $\mathbf{P}^9$  (i.e., factors through  $\pi$ ) carries a 1-parameter family of  $g_4^1$ 's.

(iv) The generic hyperplane section of  $X$  carries no  $g_4^1$ 's, but does have a unique  $g_6^2$ .

The proofs are quite straightforward: let  $C$  be a nonsingular section of  $|L|$ . The sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(L) \rightarrow \mathcal{O}_C(L) \rightarrow 0$$

gives rise to

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(X, L) \rightarrow H^0(C, \omega_C) \rightarrow 0;$$

hence

$$h^0(X, L) = 1 + h^0(C, \omega_C) = 1 + g(C) = 11,$$

where the last step follows from

$$\begin{aligned} \deg(\omega_C) &= \deg(L|_C) = \deg(L) = \deg(\pi) \cdot \deg(\mathcal{O}(3)) \\ &= 2 \cdot 3^2 = 18 \Rightarrow g(C) = 10. \end{aligned}$$

We thus have a decomposition

$$H^0(X, L) \approx \pi^*H^0(\mathbf{P}^2, \mathcal{O}(3)) \oplus R,$$

where  $R$  is the 1-dimensional subspace of  $H^0(X, L)$  consisting of sections vanishing on the ramification locus  $\pi^{-1}(B) \subset X$ . This proves claims (i) and (ii). If  $C \subset X$  comes from  $\mathbf{P}^9$  it is thus a double cover of a plane cubic  $\pi(C) \subset \mathbf{P}^2$ ; the 1-parameter family of  $g_4^1$ 's is just  $\pi^*$  of the 1-parameter family of  $g_2^1$ 's on  $\pi(C)$ . For any other hyperplane section  $C$ ,  $\pi(C)$  is a plane sextic, whence the  $g_6^2$ ; when  $C \in \mathbf{P}(R)$  is the ramification curve,  $\pi(C) = B$  is nonsingular by assumption, hence carries no  $g_4^1$ .

**(2.3) A counterexample:  $g_d^1$ 's which propagate but are not induced.** Consider  $X$  as in (2.2), but now take  $L := \pi^* \mathcal{O}_{\mathbf{P}^2}(4)$ . A computation as above shows that for generic  $C \in |L|$ ,  $g(C) = 17$  and  $\pi: X \rightarrow \mathbf{P}^2$  maps  $C$  (birationally) to a plane curve  $\pi(C)$  of degree 8, hence with  $(7 \cdot 6)/2 - 17 = 4$  nodes. We see that the generic  $C$  has a  $g_8^2$  as well as four  $g_6^1$ 's; the  $g_8^2$  is induced from a line bundle on  $X$ , but not the  $g_6^1$ 's.

Let  $P_1, P_2, P_3, P_4$  be the nodes of  $\pi(C)$ , and let  $|Z|$  be the  $g_6^1$  on  $C$  induced by the node  $P_1$ . If we fix a divisor  $Z_0 \in |Z|$  consisting of distinct points, then there is some line  $l$  in  $\mathbf{P}^2$  such that  $l \cap \pi(C) = 2P_1 + \pi(Z_0)$ ; by choosing  $Z_0$  appropriately we may assume that  $l$  does not contain  $P_i$  for  $i \neq 1$ , and that  $\pi(Z_0)$  does not contain  $P_1$ . It is easily seen that the Brill-Noether number of  $|Z|$  is  $\rho = -7 < 0$ .

Let us check that  $h^0(\mathcal{O}_C(2Z)) = 3$ ; as we shall see in the next section, this is equivalent to the  $g_6^1 |Z|$  being scheme-theoretically isolated in moduli. By duality, it suffices to check that  $h^0(\mathcal{O}_C(\omega_C - 2Z)) = 7$ .

Let  $W \in |\omega_C - 2Z_0|$ , so that  $W + 2l \in |\omega_C|$ . Then there is a plane curve  $D$  of degree 5 passing through  $P_1, P_2, P_3$ , and  $P_4$  such that

$$D \cap \pi(C) = 2 \sum_{i=1}^4 P_i + W + 2Z_0.$$

Now  $D \cap l \supset Z_0$  so that if  $l$  is not a component of  $D$  we have  $5 = D \cdot l \geq \deg Z_0 = 6$ , a contradiction. Thus,  $D = D_1 \cup l$  with  $\deg D_1 = 4$ . Since

$$D_1 \cap \pi(C) = 2P_2 + 2P_3 + 2P_4 + W + Z_0,$$

a similar argument shows that  $D_1 = D_2 \cup l$  with  $\deg D_2 = 3$ ; moreover,  $2P_1 \subset W$ . Thus,

$$D_2 \cap \pi(C) = 2P_2 + 2P_3 + 2P_4 + (W - 2P_1).$$

Moreover,  $D = D_2 \cup 2l$  passes through  $P_1, P_2, P_3$  and  $P_4$  so that  $D_2$  must pass through  $P_2, P_3$  and  $P_4$ .

We conclude that divisors in  $|\omega_C - 2Z_0|$  are in one-to-one correspondence with plane cubics passing through  $P_2, P_3$  and  $P_4$ . Since 3 points impose independent conditions on cubics (cf. Griffiths and Harris [5, p. 715]) we see that

$$h^0(\omega_C - 2Z) = 10 - 3 = 7,$$

as desired.

### 3. Linear systems on $K3$ -sections propagate

**(3.1) Theorem.** *Let  $X \subset \mathbf{P}^g$  be a  $K3$  surface, and  $C := X \cap H \subset \mathbf{P}^{g-1}$  a nonsingular hyperplane section of  $X$ . ( $C$  is canonically embedded in*

$\mathbf{P}^{g-1} \approx H$ .) If  $C$  has a  $g_d^1$  which is scheme-theoretically isolated on  $C$ , then every nonsingular hyperplane section  $C'$  of  $X$  has a  $g_d^1$ .

Let  $\mathcal{F}_d^1$  denote the space of pairs consisting of a curve  $C$  and a  $g_d^1$  on it, let  $\mathcal{M}_d^1 \subset \mathcal{M}_g$  be the space of  $d$ -gonal curves, and for fixed  $C \in \mathcal{M}_d^1$  let  $W_d^1$  denote the fiber of  $\mathcal{F}_d^1$  over  $C$ . We recall that the  $g_d^1 |Z|$  on  $C$  is scheme-theoretically isolated if

$$T_{|Z|}W_d^1 = (0).$$

Equivalently,  $\mathcal{F}_d^1$  must be transversal to the Jacobian of  $C$ . We have:

- $H^0(\omega - 2Z)$  injects into  $H^0(\omega^2)$ , and the image can be naturally identified with the conormal space at  $C$  to the local component of  $\mathcal{M}_d^1$  corresponding to  $(C, |Z|)$ ,

- $\dim \mathcal{F}_d^1 = 2g + 2d - 5$ .

Putting these together, we see that the transversality is equivalent to

$$h^0(\omega - 2Z) + (2g + 2d - 5) = 3g - 3,$$

or

$$h^0(\omega - 2Z) = g - 2d + 2,$$

and by Serre duality, to

$$h^0(2Z) = 3.$$

Our theorem thus follows from the following more general statement:

**(3.2) Theorem.** *Let  $X \subset \mathbf{P}^g$  be a K3 surface,  $C_0 := X \cap H_0$  a nonsingular hyperplane section, and  $|Z|$  a  $g_d^r$  on  $C_0$  which is scheme-theoretically isolated on  $C_0$ , and satisfies*

$$h^0(C_0, \mathcal{O}(2Z)) = 2r + 1.$$

*Then every nonsingular hyperplane section  $C$  of  $X$  has a  $g_d^r$ .*

**(3.3) Iterative construction.** We construct a series of subvarieties  $\mathcal{H}_i \subset (\mathbf{P}^g)^*$ ,  $\mathcal{S}_i \subset S^d(X)$ , and correspondences  $\mathcal{F}_i, \mathcal{F}'_i \subset (\mathbf{P}^g)^* \times S^d(X)$ , as follows. Let  $\mathcal{S}_0 := \{Z_0\}$  for some fixed divisor  $Z_0 \in |Z|$  consisting of distinct points. Define inductively, for  $i \geq 1$ :

$$\mathcal{F}'_i := \{(Z, H) \in \mathcal{S}_{i-1} \times (\mathbf{P}^g)^* \mid H \supset \text{span}(Z)\},$$

$\mathcal{F}_i :=$  unique irreducible component of  $\mathcal{F}'_i$  which dominates  $\mathcal{S}_{i-1}$ ,

$$\mathcal{H}_i := \text{pr}_2(\mathcal{F}_i) \subset (\mathbf{P}^g)^*,$$

$$\mathcal{F}_i := \left\{ (Z, H) \mid \begin{array}{l} H \in \mathcal{H}_i, Z \in S^d C \text{ where } C := X \cap H \\ \exists Z' \in S^d C \text{ such that } (Z', H) \in \mathcal{F}_i \text{ and } Z \sim_C Z' \end{array} \right\},$$

where " $\sim_C$ " means linear equivalence on  $C$ ,

$$\mathcal{S}_i := \text{pr}_1(\mathcal{F}_i) \subset S^d(X).$$

We note that for all  $(Z, H) \in \mathcal{F}_i$ ,

$$h^0(X \cap H, \mathcal{O}(Z)) = r + 1.$$

This is an easy induction, based on the observation that the left-hand side depends, by the geometric version of Riemann-Roch, only on the position of the  $d$ -tuple  $Z$  in  $\mathbf{P}^g$  and not on the choice of canonical curve through these points. Hence  $\mathcal{F}_i$  is dominated by a  $\mathbf{P}^r$ -bundle over  $\mathcal{F}_i$ , so another easy induction shows that  $\mathcal{F}_i$  is irreducible. (Actually, the same argument shows that  $\mathcal{F}'_i = \mathcal{F}_i$  is already irreducible.)

Consider the following diagrams:

$$\begin{array}{ccc} & \mathcal{F}_i & \\ g-d+r \swarrow & & \searrow \\ \mathcal{F}_{i-1} & & \mathcal{H}_i \end{array} \qquad \begin{array}{ccc} & \mathcal{F}'_i & \\ \swarrow & & \searrow r \\ \mathcal{F}_i & & \mathcal{H}_i \end{array}$$

What we know about them can be summarized as follows:

- (1) All four maps are surjective.
- (2) All fibers of  $\text{pr}_1: \mathcal{F}_i \rightarrow \mathcal{F}_{i-1}$  are  $(g-d+r)$ -dimensional.
- (3) All fibers of  $\text{pr}_2: \mathcal{F}'_i \rightarrow \mathcal{H}_i$  are at least  $r$ -dimensional; the fiber over  $H_0$  has an irreducible component which is precisely  $r$ -dimensional, by our assumption that  $Z_0$  is isolated.

The sequences  $\mathcal{F}_i, \mathcal{F}'_i, \mathcal{H}_i, \mathcal{S}_i$  stabilize for large  $i$ , and we let  $\mathcal{F} = \mathcal{F}_i$ ,  $\mathcal{H}$  and  $\mathcal{S}$  denote the respective limits. From the diagrams we have:

$$\dim(\mathcal{S}) + g - d + r = \dim(\mathcal{F}) = \dim(\mathcal{F}') = \dim(\mathcal{H}) + r,$$

where the last step follows from (3) above together with the irreducibility of  $\mathcal{F}'$ . Our theorem that  $\dim(\mathcal{H}) = g$ , is thus equivalent to  $\dim(\mathcal{S}) = d$ . In fact, we claim that already

$$\dim(\mathcal{S}_1) = d.$$

Indeed,  $\text{span}(Z_0)$  is a  $\mathbf{P}^{d-r-1}$ , i.e. contained in a  $(g-d+r)$ -dimensional family of hyperplanes, i.e.  $\dim(\mathcal{H}_1) = g-d+r$ . Therefore,

$$\dim(\mathcal{F}_1) = g - d + 2r.$$

By the geometric version of Riemann-Roch, our assumption  $h^0(C_0, \mathcal{O}(2Z)) = 2r + 1$  is equivalent to saying that for  $Z_1 \neq Z_0$ ,  $\text{span}(Z_0, Z_1)$  is a  $\mathbf{P}^{2d-2r-1}$ . Hence the fibers of  $\text{pr}_1: \mathcal{F}_1 \rightarrow \mathcal{S}_1$  have dimension  $g - 2d + 2r$ , so

$$\dim(\mathcal{S}_1) = (g - d + 2r) - (g - 2d + 2r) = d$$

as claimed. This proves Theorems (3.1) and (3.2).

#### 4. Constancy of the Clifford index

Our main result in this section is a proof of Green's conjecture (1.1) for  $g_d^1$ 's.

**(4.1) Theorem.** *Let  $C$  be a nonsingular curve of genus  $g \geq 2$  on a K3 surface  $X$ , and suppose there is a  $g_d^1 |Z|$  on  $C$  achieving the Clifford index,  $\nu(C) = d - 2$ . Then  $\nu(C) = \nu(\mathcal{O}_X(C))$ .*

In view of the semicontinuity of the Clifford index, it will suffice to prove a particular case of conjecture (1.2): that there is a linear system on  $X$  whose restriction to  $C$  contains  $|Z|$  and whose restriction to any  $C' \in |C|$  has the same Clifford index as  $|Z|$ .

**(4.2) Theorem.** *Under the assumptions of (4.1), there is a divisor  $D \subset X$  such that*

- $\nu(Z) = \nu(C) = \nu(\mathcal{O}_C(D))$ .
- $h^0(\mathcal{O}_X(D)) \geq 2$ ,  $h^0(\mathcal{O}_X(C - D)) \geq 1$ ,  $\deg(\mathcal{O}_C(D)) \leq g - 1$ .
- *There is some  $Z_0 \in |Z|$ , consisting of distinct points, such that  $Z_0 \subset D \cap C$ .*
- *For nonsingular  $C' \in |C|$ ,  $\nu(\mathcal{O}_{C'}(D)) = \nu(\mathcal{O}_C(D))$ ,  $h^0(\mathcal{O}_{C'}(D)) \geq 2$  and  $\deg(\mathcal{O}_{C'}(D)) \leq g - 1$ .*

There are two easy reduction steps in the proof of this theorem. First, we may assume that  $C$  is nonhyperelliptic (since the hyperelliptic case is covered by [14]), and hence that  $\varphi_{|C|}$  is birational, and its restriction to  $C$  embeds  $C$  as a canonical curve. Second, notice that  $|Z|$  is base-point-free and complete (else there would be a  $g_{d'}^1$  or  $g_d^r$  with Clifford index  $d' - 2 < d - 2$  or  $d - 2r < d - 2$ ).

In §5, we will extend (4.2) to  $g_d^1$ 's which do not necessarily achieve the Clifford index. We therefore state our hypotheses explicitly, so that our lemmas can be reused in §5. We assume only:

- $C$  is a nonsingular nonhyperelliptic curve of genus  $g \geq 2$ .
- $|Z|$  is a complete base-point free  $g_d^1$  on  $C$ , and a divisor  $Z_0 \in |Z|$  has been chosen, consisting of distinct points none of which lies on any (of the countably many) rational curves on  $X$ .
- The Brill-Noether number  $\rho(Z) = 2d - 2 - g$  is negative.

Our first lemma was inspired by work of Lazarsfeld and Reider.

**(4.3) Lemma.** *Under our hypotheses, there is a rank-2, nonsimple vector bundle  $\mathcal{F} \rightarrow X$  with  $c_1(\mathcal{F}) = [C]$  and  $c_2(\mathcal{F}) = d$ , and a section  $s$  of  $\mathcal{F}$  with  $(s) = Z_0$ .*

*Proof.* We use a construction of Griffiths and Harris, Proposition (1.33) in [6]. This provides  $\mathcal{F}$  and  $s$  with the required invariants; the condition needed

is that any divisor in  $|C|$  which passes through all-but-one points of  $Z_0$  must pass through the remaining point. By surjectivity of

$$H^0(X, \mathcal{O}_X(C)) \rightarrow H^0(C, \omega_C)$$

we are reduced to the same condition for  $Z_0$  and the canonical system,  $|\omega_C|$ . By Riemann-Roch, this is equivalent to our assumptions that  $\dim |Z| > 1$  and that  $|Z|$  is base-point-free.

We still need to check that  $\mathcal{F}$  is nonsimple, i.e. that  $h^0(\mathcal{F} \otimes \mathcal{F}^*) > 1$ . But this is a straightforward computation (cf. Lazarsfeld [8] and Mukai [10]):

$$\begin{aligned} \chi(\mathcal{F} \otimes \mathcal{F}^*) &= c_1^2(\mathcal{F}) - 4c_2(\mathcal{F}) + 4\chi(\mathcal{O}_X) \\ &= 2g - 2 - 4d + 8 = -2\rho(Z) + 2 > 2, \end{aligned}$$

but since  $\mathcal{F} \otimes \mathcal{F}^*$  is self dual,

$$\chi(\mathcal{F} \otimes \mathcal{F}^*) = 2h^0(\mathcal{F} \otimes \mathcal{F}^*) - h^1(\mathcal{F} \otimes \mathcal{F}^*)$$

so we conclude  $h^0(\mathcal{F} \otimes \mathcal{F}^*) > 1$ .

**Remarks.** (i) The bundle  $\mathcal{F}$  in (4.3) is the dual of the one constructed by Lazarsfeld [8].

(ii) Reider's method [13] is as follows: the computation above shows that  $c_1^2(\mathcal{F}) > 4c_2(\mathcal{F})$  exactly when  $\rho(Z) < -3$ . In that case, a theorem of Bogomolov [2] yields the conclusion in case (a) of Lemma (4.4) below.

**(4.4) Lemma.** *Let  $\mathcal{F}$  be a nonsimple, rank-2 vector bundle on  $X$ . There exist line bundles  $L, M$  and a zero-dimensional subscheme  $A \subset X$  such that  $\mathcal{F}$  fits in an exact sequence*

$$0 \rightarrow L \rightarrow \mathcal{F} \xrightarrow{\pi} M \otimes \mathcal{I}_A \rightarrow 0$$

and either

- (a)  $L \geq M$ , or
- (b)  $A$  is empty and the sequence splits,  $\mathcal{F} \approx L \oplus M$ .

*Proof.* Since  $\mathcal{F}$  is nonsimple, a standard argument shows the existence of an endomorphism  $\varphi: \mathcal{F} \rightarrow \mathcal{F}$  which drops rank everywhere.<sup>1</sup> Let  $L, N$  be the kernel and image of  $\varphi$  respectively, and  $M := N^{**}$ , the double dual. Clearly,  $L$  and  $M$  are line bundles and  $N = \mathcal{I}_A \otimes M$  for some zero-dimensional  $A \subset X$ .

The two cases arise as follows: if  $\varphi^2 = 0$ , then  $N = \text{im}(\varphi) \subset \ker(\varphi) = L$ , so  $L \otimes M^{-1} \approx L \otimes N^*$  has a section, and we are in case (a). Otherwise,  $\varphi$

<sup>1</sup>If  $\mathcal{F}$  is decomposable, take  $\varphi$  to be projection onto a summand. If  $\mathcal{F}$  is indecomposable, let  $\varphi_0$  be any automorphism of  $\mathcal{F}$  which is not a multiple of the identity  $1_{\mathcal{F}}$ , and let  $\lambda$  be an eigenvalue of  $\varphi_0$  at any point. Then  $\varphi := \varphi_0 - \lambda 1_{\mathcal{F}}$  is not an automorphism, so by a theorem of Atiyah [1], it must be nilpotent; since  $\varphi \neq 0$ , it must drop rank everywhere.

must induce an isomorphism from  $N$  to its image in  $\mathcal{F}$ , thus splitting the sequence

$$0 \rightarrow L \rightarrow \mathcal{F} \rightarrow N \rightarrow 0.$$

Since  $\mathcal{F}$  is locally free,  $N$  must be a line bundle, i.e.,  $A = \emptyset$  and we are in case (b).

**(4.5) Corollary.** *Under our hypotheses, there exist effective divisors  $D, \Delta$  on  $X$  such that  $C \sim D + \Delta$ ,  $Z_0 \subset D \cap \Delta$ ,  $D \cdot \Delta = d - \deg(A)$ , and either*

*(Case (a))  $\Delta - D$  is effective, or*

*(Case (b))  $D$  meets  $\Delta$  transversally and  $Z_0 = D \cap \Delta$ .*

*Proof.* We apply (4.4) to (4.3). The section  $s \in H^0(\mathcal{F})$  vanishes on the 0-dimensional locus  $Z_0$ , hence is not contained in the line-subbundle  $L$ . The projection  $\pi(s)$  is therefore a nonzero section of  $M \otimes \mathcal{F}_A$ ; let  $D$  be its 0-locus, so

$$M \approx \mathcal{O}_X(D), \quad Z_0 \subset D.$$

In case (a) we take  $\Delta = D + E$ , where  $E$  is an effective divisor in  $|L \otimes M^{-1}|$ , so that  $L \approx \mathcal{O}_X(\Delta)$ , and we have

$$Z_0 \subset D = D \cap (D + E) = D \cap \Delta$$

and

$$d - \deg(A) = c_2(\mathcal{F}) - \deg(A) = D \cdot \Delta.$$

In case (b) we have a decomposition  $s = s_L \oplus s_M$ , so we define

$$D := (s_M), \quad \Delta := (s_L).$$

Then  $Z_0$  equals the intersection, which must be transversal since  $Z_0$  consists of distinct points.

**(4.6) Lemma.** *Under our hypotheses,  $\nu(\mathcal{O}_C(D)) \leq \nu(Z)$ .*

*Proof.*

$$\begin{aligned} \nu(\mathcal{O}_C(D)) &= C \cdot D - 2h^0(\mathcal{O}_C(D)) + 2 \\ &\leq C \cdot D - 2h^0(\mathcal{O}_X(D)) + 2 \\ &\leq C \cdot D - (D \cdot D + 4) + 2 = \Delta \cdot D - 2 \\ &= d - \deg(A) - 2 \leq d - 2 = \nu(Z). \end{aligned}$$

The first inequality follows from the exact sequence

$$0 \rightarrow \mathcal{O}_X(-\Delta) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0,$$

and the second from Riemann-Roch for the line bundle  $\mathcal{O}_X(D)$ . q.e.d.

The proofs of Theorems (4.1) and (4.2) can now be completed: the extra hypothesis is that  $\nu(Z)$  is minimal, so the inequality in (4.6) must be an equality. In particular, we must have:

- (1)  $H^0(\mathcal{O}_X(D)) \xrightarrow{\sim} H^0(\mathcal{O}_C(D))$  is an isomorphism;
- (2)  $H^1(\mathcal{O}_X(D)) = 0$
- (3)  $A = \emptyset$ .

Combining (1) and (2) we get  $H^1(D-C) = 0$ . But then also  $H^1(D-C') = 0$  for  $C' \in |C|$ , so we get an isomorphism:

$$H^0(\mathcal{O}_X(D)) \xrightarrow{\sim} H^0(\mathcal{O}_{C'}(D))$$

for nonsingular  $C' \in |C|$ , so finally

$$h^0(\mathcal{O}_{C'}(D)) = h^0(\mathcal{O}_C(D))$$

as required.

### 5. Linear systems on K3-sections are contained in induced ones

**(5.1) Theorem.** *Let  $X$  be a K3 surface,  $C \subset X$  a nonsingular, nonhyperelliptic curve, and  $|Z|$  a complete, base-point-free  $g_d^1$  on  $C$  with  $\rho(\mathcal{O}_C(Z)) < 0$ . Then there is a line bundle  $L \rightarrow X$  such that*

- $h^0(X, L) \geq 2$ ,  $h^0(X, \mathcal{O}_X(C) \otimes L^{-1}) \geq 2$ ,  $\deg(L \otimes \mathcal{O}_C) \leq g - 1$ .
- $\nu(\mathcal{O}_C \otimes L) \leq \nu(\mathcal{O}_C(Z))$ .
- $\nu(\mathcal{O}_{C'} \otimes L) = \nu(\mathcal{O}_C \otimes L)$  for nonsingular  $C' \in |C|$ .

• There are divisors  $Z_0 \in |Z|$  (consisting of distinct points) and  $D \in |L|$  such that  $Z_0 \subset D \cap C$ .

For the proof we use the techniques of §4, with one new idea. The problem is that even after we have manufactured the splitting  $C \sim D + \Delta$ , we are not done: the inequalities in (4.6) may not be equalities, so  $H^0(\mathcal{O}_C(D))$  may be bigger than  $H^0(\mathcal{O}_X(D))$ , and no conclusion can be made about  $\nu(\mathcal{O}_{C'}(D))$ .

We thus introduce a definition: a line bundle  $L = \mathcal{O}_X(D)$  is *adapted to  $|C|$*  if

- (1)  $h^0(\mathcal{O}_X(D)) \geq 2$ ,  $h^0(\mathcal{O}_X(C - D)) \geq 2$ , and
- (2)  $h^0(\mathcal{O}_{C'}(D))$  is independent of the nonsingular  $C' \in |C|$ .

The theorem can thus be rephrased:

**(5.1') Theorem.** *Let  $X$  be a K3 surface,  $C \subset X$  a nonsingular, nonhyperelliptic curve, and  $|Z|$  a complete, base-point-free  $g_d^1$  on  $C$  with  $\rho(\mathcal{O}_C(Z)) < 0$ . Then there is a line bundle  $L \rightarrow X$  adapted to  $|C|$  such that*

- $\nu(L \otimes \mathcal{O}_C) \leq \nu(Z)$ .
- For some divisors  $Z_0 \in |Z|$  (distinct points) and  $D \in |L|$ ,  $Z_0 \subset D \cap C$ .

**(5.2) Lemma.**  *$L = \mathcal{O}_X(D)$  is adapted to  $|C|$  if*

- (1)  $h^0(\mathcal{O}_X(D)) \geq 2$ ,  $h^0(\mathcal{O}_X(C - D)) \geq 2$ , and
- (2') Either  $h^1(\mathcal{O}_X(D)) = 0$  or  $h^1(\mathcal{O}_X(C - D)) = 0$ .

*Proof.* The sheaf sequence

$$0 \rightarrow \mathcal{O}_X(D - C') \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_{C'}(D) \rightarrow 0$$

gives

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}_X(D - C')) \rightarrow H^0(\mathcal{O}_X(D)) \rightarrow H^0(\mathcal{O}_{C'}(D)) \\ \rightarrow H^1(\mathcal{O}_X(D - C')) \xrightarrow{\alpha} H^1(\mathcal{O}_X(D)). \end{aligned}$$

We note that

$$h^i(\mathcal{O}_X, D - C') = h^i(\mathcal{O}_X, D - C)$$

is independent of  $C'$ . Hence  $h^0(\mathcal{O}_{C'}(D))$  is determined by  $\text{rank}(\alpha)$ ; the alternatives in (2') assure  $\text{rank}(\alpha) = 0$ . (Note that  $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D - C)) = h^1(\mathcal{O}_X(D - C'))$ .)

**(5.3) Proposition.** *Let  $D$  be a divisor on  $X$  such that  $h^0(\mathcal{O}_X(D)) \geq 2$  and  $h^0(\mathcal{O}_X(C - 2D)) \geq 1$ . Then there is a divisor  $\tilde{D}$  on  $X$  such that*

(i)  $\mathcal{O}_X(\tilde{D})$  is adapted to  $|C|$ .

(ii)  $h^0(\mathcal{O}_X(C - 2\tilde{D})) \geq 1$ .

(iii)  $\tilde{D} \cdot (C - \tilde{D}) \leq D \cdot (C - D)$ .

(iv) For some  $\Gamma_0$  which is either empty or a smooth rational curve,  $D - \tilde{D} + \Gamma_0$  is an effective divisor whose support is a union of smooth rational curves.

*Proof.* Let  $E$  be an effective divisor in the linear system  $|C - 2D|$ . We apply (2.1) to  $\mathcal{O}_X(D)$ .

Suppose first that  $D$  is nef. If  $D^2 > 0$  then  $h^1(\mathcal{O}_X(D)) = 0$  and  $\mathcal{O}_X(D)$  is adapted to  $|C|$  by Lemma (5.2); set  $\tilde{D} := D$ . Otherwise,  $D^2 = 0$  and  $\mathcal{O}_X(D)$  has the type of example X1, that is,  $D \sim kF$  for some smooth elliptic curve  $F$ . If  $k = 1$ , then  $h^1(\mathcal{O}_X(D)) = 0$  so  $\mathcal{O}_X(D)$  is still adapted to  $|C|$  and we may set  $\tilde{D} = D$ .

Thus, we may assume  $D \sim kF$  with  $k \geq 2$ . We now apply (2.1) to  $\mathcal{O}_X(D + E)$ . If  $D + E$  is not nef, let  $\Gamma_0$  be a smooth rational curve such that  $(D + E) \cdot \Gamma_0 < 0$ , and let  $\tilde{D} := D + \Gamma_0 \sim kF + \Gamma_0$ . We claim that  $\tilde{D}$  is nef: the only curve which could possibly have negative intersection number with  $\tilde{D}$  is  $\Gamma_0$ , but

$$F \cdot \Gamma_0 = \frac{1}{k} D \cdot \Gamma_0 = \frac{1}{k} (C \cdot \Gamma_0 - (D + E) \cdot \Gamma_0) \geq -\frac{1}{k} (D + E) \cdot \Gamma_0 > 0,$$

so that

$$\tilde{D} \cdot \Gamma_0 = kF \cdot \Gamma_0 - 2 \geq k - 2 \geq 0.$$

Thus  $\tilde{D}$  is nef: moreover,  $\tilde{D}^2 = (kF + \Gamma_0)^2 = 2kF \cdot \Gamma_0 - 2 \geq 2k - 2 \geq 2$ , so that  $h^1(\mathcal{O}_X(\tilde{D})) = 0$  by (2.1). Hence  $\mathcal{O}_X(\tilde{D})$  is adapted to  $|C|$ .

We must check the other properties claimed for  $\tilde{D}$  in this case. Since

$$E \cdot \Gamma_0 = (D + E) \cdot \Gamma_0 - kF \cdot \Gamma_0 \leq (D + E) \cdot \Gamma_0 - k < -k \leq -2,$$

we have  $E - \Gamma_0$  effective. Furthermore,

$$(E - \Gamma_0) \cdot \Gamma_0 = E \cdot \Gamma_0 + 2 < 0,$$

so that  $E - 2\Gamma_0$  is effective as well. Thus,

$$h^0(\mathcal{O}_X(C - 2\tilde{D})) = h^0(\mathcal{O}_X(E - 2\Gamma_0)) \geq 1,$$

verifying property (ii). Property (iv) is clear from the definition of  $\tilde{D}$ ; to check property (iii), we compute

$$\begin{aligned} \tilde{D} \cdot (C - \tilde{D}) &= D \cdot (C - D) + \Gamma_0 \cdot (C - 2D) - \Gamma_0^2 \\ &= \tilde{D} \cdot (C - D) + \Gamma_0 \cdot E + 2 \leq D \cdot (C - D). \end{aligned}$$

To complete the proof in the case that  $D$  is nef, we may thus assume  $D \sim kF$  with  $k \geq 2$  and  $D + E$  is nef. If  $(D + E)^2 > 0$ , then by (2.1),

$$h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D + E)) = 0$$

so that  $\mathcal{O}_X(D)$  is once again adapted to  $|C|$  by (5.2), and we may set  $\tilde{D} := D$ . Otherwise,  $D + E \sim lG$  for some smooth elliptic curve  $G$ , and every divisor in  $|D + E|$  has the form  $G_1 + \cdots + G_l$  for certain  $G_i \in |G|$ . Since  $kF + E \in |D + E|$ , we must in fact have  $|F| = |G|$ . But then  $C \sim (k + l)F$  so that  $C^2 = 0$ , a contradiction.

To prove the proposition in general, we use induction on the number of base components of  $|D|$ , counted with multiplicity. If  $|D|$  has no base components then  $D$  is nef and we are finished. If  $|D|$  has  $m$  base components, we may assume that  $D$  is not nef (else we are finished as above) and let  $\Gamma$  be a smooth rational curve with  $D \cdot \Gamma < 0$ . Then  $\Gamma$  is a base component of  $|D|$ , and  $|D - \Gamma|$  has  $m - 1$  base components. By inductive hypothesis, there is a  $\tilde{D}$  adapted to  $|C|$  with  $h^0(\mathcal{O}_X(C - 2\tilde{D})) \geq 1$  such that  $\tilde{D} \cdot (C - \tilde{D}) \leq (D - \Gamma) \cdot (C - D + \Gamma)$  and  $(D - \Gamma) - \tilde{D} + \Gamma_0$  is effective and supported on rational curves for some  $\Gamma_0$ . Since  $D - \tilde{D} + \Gamma_0 = ((D - \Gamma) - \tilde{D} + \Gamma_0) + \Gamma$ , it suffices to show that

$$(D - \Gamma) \cdot (C - D + \Gamma) \leq D \cdot (C - D),$$

i.e., since  $(D - \Gamma) \cdot (C - D + \Gamma) = D \cdot (C - D) - \Gamma \cdot E + 2$  it suffices to show that  $\Gamma \cdot E \geq 2$ . But  $\Gamma \cdot D \leq -1$  so that

$$\Gamma \cdot E = \Gamma \cdot C - 2\Gamma \cdot D \geq -2\Gamma \cdot D \geq 2. \quad \text{q.e.d.}$$

We can now complete the proof of (5.1). We choose  $Z_0 \in |Z|$  as in §4, consisting of distinct points not on any nonsingular rational curve in  $X$ . We apply (4.5) to obtain  $D, \Delta$ , with  $D \cap \Delta \supset Z_0$ .

In case (a) of (4.5), we use Proposition (5.3) to replace  $D$  by  $\tilde{D}$  which is adapted to  $|C|$ , with

$$\tilde{D} \cdot (C - \tilde{D}) \leq D \cdot (C - D) = D \cdot \Delta.$$

Since  $D - \bar{D} + \Gamma_0$  is supported on rational curves, it does not meet  $Z_0$ , so  $Z_0 \subset D \Rightarrow Z_0 \subset \bar{D}$ . We now apply Lemma (4.6) to  $\bar{D}$ , concluding that  $\nu(\mathcal{O}_C(\bar{D})) \leq \nu(Z)$ . We may thus take  $L := \mathcal{O}_X(\bar{D})$ .

In case (b) of (4.5), we simply take  $L := \mathcal{O}_X(D)$ . We claim:

$$\begin{aligned} h^1(\mathcal{O}_X(D)) &= h^1(\mathcal{O}_X(\Delta)) = 0, \\ h^0(\mathcal{O}_X(D)) &\geq 2, \quad h^0(\mathcal{O}_X(\Delta)) \geq 2. \end{aligned}$$

By symmetry, it suffices to check this for  $D$ . We use the results of (2.1):

If  $D$  is not nef: there is a smooth, rational  $\Gamma$  such that  $D \cdot \Gamma < 0$ , so  $D_0 := D - \Gamma$  is effective. We have

$$\Gamma \cdot \Delta = \Gamma \cdot (C - D) = \Gamma \cdot C - \Gamma \cdot D > 0 - 0 = 0,$$

so  $Z_0 = D \cap \Delta \supset \Gamma \cap \Delta$  must contain a point of  $\Gamma$ , a contradiction.

If  $D^2 > 0$  then  $h^1(\mathcal{O}_X(D)) = 0$ ,  $h^0(\mathcal{O}_X(D)) \geq 2$  and we are done. By (2.1), the only remaining case is  $X1$ :

$$D \sim kF, \quad F \text{ nonsingular elliptic, } k \geq 1,$$

and then

$$h^0(L) = k + 1, \quad h^1(L) = k - 1.$$

We claim that  $k = 1$ . Indeed,

$$D \cdot C = D \cdot (C - D) = D \cdot \Delta = d,$$

so  $Z_0 = D \cap C$ , hence

$$2 = h^0(\mathcal{O}_C(Z)) = h^0(\mathcal{O}_C(D)) \geq h^0(\mathcal{O}_X(D)) \geq 2$$

so

$$k + 1 = h^0(\mathcal{O}_X(D)) = 2$$

as required.

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