Algebraic cycles on products of surfaces

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Throughout this note, an algebraic surface means a smooth projective variety of dimension 2 over the complex numbers $\mathbb{C}$. An algebraic surface X has a Néron-Severi group NS(X), which is the group of cohomology classes of line bundles on X, and is naturally a subgroup of the second cohomology $H^2(X,\mathbb{Z})$. The group NS(X)/torsion has a natural integer-valued quadratic form induced by the intersection pairing on $H^2(X,\mathbb{Z})$. We let $NS(X,\mathbb{Q})$ denote $NS(X)\otimes\mathbb{Q}$, and let $\rho(X) = \dim_{\mathbb{Q}} NS(X,\mathbb{Q})$ be the Picard number of X. By a theorem of Lefschetz and Hodge [1], $NS(X,\mathbb{Q}) = H^{1,1}(X)\cap H^2(X,\mathbb{Q})$.

An algebraic surface X also has a transcendental lattice $T(X,\mathbb{Z})$; this is defined to be the cokernel $H^2(X,\mathbb{Z})/NS(X)$, and may be identified with the orthogonal complement of NS(X)/torsion inside $H^2(X,\mathbb{Z})$/torsion. With this second interpretation, $T(X,\mathbb{Z})$ also inherits an integer-valued quadratic form from the intersection form on $H^2(X,\mathbb{Z})$. We let $T(X,\mathbb{Q})$ denote $T(X,\mathbb{Z})\otimes\mathbb{Q}$. Note that $T(X,\mathbb{Z})$ and $T(X,\mathbb{Q})$ are birational invariants of X.

If Z is any smooth complex projective variety, the cup product gives the rational cohomology $\oplus H^i(Z,\mathbb{Q})$ the structure of a $\mathbb{Q}$-algebra. Let Hdg(Z) be the $\mathbb{Q}$-subalgebra of $\oplus H^i(Z,\mathbb{Q})$ defined by

$$\mathrm{Hdg}(Z) = \bigoplus_p H^{p,p}(Z)\cap H^{2p}(Z,\mathbb{Q}).$$

Theorem (Hodge [1]). If $W \subset Z$ is an algebraic cycle of codimension p, then the cohomology class of W lies in $\mathrm{Hdg}(Z) \cap H^{2p}(Z,\mathbb{Q})$.

Conjecture (Hodge [1]). Every element of Hdg(Z) is the cohomology class of a $\mathbb{Q}$-linear combination of algebraic cycles. (An excellent account of the current status of this conjecture can be found in [17].)

Let Div(Z) be the $\mathbb{Q}$-subalgebra of Hdg(Z) generated by the fundamental class of Z in $H^0(Z,\mathbb{Q})$ and by the classes of divisors on Z; that is, by $\mathrm{Hdg}(Z) \cap (H^0(Z,\mathbb{Q}) \oplus H^2(Z,\mathbb{Q}))$. (Elements of Div(Z) are $\mathbb{Q}$-linear combinations of classes of complete intersections on Z.)

Definition (Okamoto [13]). Let X and Y be algebraic surfaces. Define the Hodge-Künneth-Transcendence group of X and Y by

$$\mathrm{HKT}(X,Y) = \mathrm{Hdg}(X \times Y) \cap (T(X,\mathbb{Q}) \otimes T(Y,\mathbb{Q})),$$

where we regard $T(X,\mathbb{Q}) \otimes T(Y,\mathbb{Q})$ as a subgroup of $H^4(X \times Y,\mathbb{Q})$ via the Künneth decomposition.

The usefulness of this definition is shown by the following lemma.

Lemma (Lieberman [4], Okamoto [13]). Let X and Y be algebraic surfaces. For each $\alpha \in \mathrm{Hdg}(X \times Y)$, there exists some $\beta \in \mathrm{Div}(X \times Y)$ (depending on α) such that $\alpha - \beta \in \mathrm{HKT}(X,Y)$.

The proof follows immediately from the following fact:

$$(*)\qquad \mathrm{Hdg}(X\times Y) \cap (H^i(X,\mathbb{Q})\otimes H^j(Y,\mathbb{Q})) = \begin{cases} (NS(X,\mathbb{Q})\otimes NS(Y,\mathbb{Q})) \oplus HKT(X,Y) & \text{if } i = j = 2 \\ \mathrm{Div}(X\times Y) \cap (H^i(X,\mathbb{Q})\otimes H^j(Y,\mathbb{Q})) & \text{otherwise.} \end{cases}$$

This is straightforward to check, except in the cases $|i-j| = 2$, $i+j = 4$: in those cases, one must use the hard Lefschetz theorem on X or Y. For example, if $i = 1$ and $j = 3$, let λ be the class of an ample divisor on Y, and let $[X]$ be the fundamental class of X in $H^0(X,\mathbb{Q})$. Then by hard Lefschetz, cup product with $[X]\times\lambda$ gives an isomorphism

$$H^1(X,\mathbb{Q})\otimes H^1(Y,\mathbb{Q}) \xrightarrow{\sim} H^1(X,\mathbb{Q})\otimes H^3(Y,\mathbb{Q})$$

which preserves the Hodge structure (with a shift in weight). Thus, for any $\alpha \in H^1(X,\mathbb{Q})\otimes H^3(Y,\mathbb{Q})$ there is some γ lying in

$$\mathrm{Hdg}(X\times Y) \cap (H^1(X,\mathbb{Q})\otimes H^1(Y,\mathbb{Q})) \subset \mathrm{Div}(X\times Y)$$

such that $\alpha = ([X]\times\lambda)\cup\gamma$. But now $[X]\times\lambda \in \mathrm{Div}(X\times Y)$, which implies that $\alpha \in \mathrm{Div}(X\times Y)$. This proves fact (*), and hence the lemma.

It is important to keep in mind that, in spite of this lemma, $\mathrm{Div}(X\times Y)\cap HKT(X,Y)$ may contain non-zero elements. An important example of this phenomenon is given by the following theorem, which seems to have been known to Mumford (cf. [9; p. 349]), and which follows from recent work of Tanke'ev [19], [20], Ribet [15], and Murty [10], [11].

<u>Theorem</u>. Let A be an abelian surface. Then $\mathrm{Div}(A\times A) = \mathrm{Hdg}(A\times A)$.

In particular, since the quadratic form on $T(A,\mathbb{Q})$, when regarded as an element of $T(A,\mathbb{Q})\otimes T(A,\mathbb{Q})$, lies in $\mathrm{HKT}(A,A)$, we see that $\mathrm{Div}(A\times A)\cap \mathrm{HKT}(A,A) \neq (0)$.

<u>Definition</u>. Let X and Y be algebraic surfaces.

(1) A <u>cohomological isogeny</u> between X and Y is an isomorphism of rational Hodge structures $T(X,\mathbb{Q}) \xrightarrow{\sim} T(Y,\mathbb{Q})$. (Note that using the intersection pairing to give an isomorphism of $T(Y,\mathbb{Q})$ with $T(Y,\mathbb{Q})^*$, we may regard a cohomological isogeny as an element of $\mathrm{HKT}(X,Y)$.)

(2) An <u>isogeny</u> between X and Y is an algebraic cycle $Z \subset X\times Y$ of codimension two which induces a cohomological isogeny (in other words, the HKT part of its cohomology class is a cohomological isogeny).

(3) An isogeny or cohomological isogeny is <u>strict</u> if it maps the intersection pairing on X to the intersection pairing on Y.

(4) An isogeny or cohomological isogeny is <u>integral</u> if it is compatible with an isomorphism of integral Hodge structures $T(X,\mathbb{Z}) \xrightarrow{\sim} T(Y,\mathbb{Z})$.

<u>Warning</u>: The term "isogeny", when applied in particular to K3 surfaces, has several conflicting definitions in the literature. Let X and Y be algebraic K3 surfaces. Inose [2] uses the term "isogeny" to mean a rational map of finite degree between X and Y (when $\rho(X) = \rho(Y) = 20$). Shafarevich [16] uses the term "isogeny" to mean an isomorphism $H^2(X,\mathbb{Q}) \xrightarrow{\sim} H^2(Y,\mathbb{Q})$ preserving the Hodge structures and the intersection

pairings. Mukai [8] uses the term "isogeny" to mean an algebraic cycle on $X \times Y$ whose cohomology class is a Shafarevich-isogeny. In our terminology, an Inose-isogeny is an isogeny, a Mukai-isogeny is a strict isogeny, and a Shafarevich-isogeny induces a strict cohomological isogeny.

We now restrict our attention to the case of algebraic surfaces with geometric genus 1. If X is such a surface, then the rational Hodge structure on $T(X,\mathbb{Q})$ is indecomposable over $\mathbb{Q}$. (For suppose there were some non-trivial decomposition $T(X,\mathbb{Q}) = T_1 \oplus T_2$; since $\dim(H^{2,0}(X)) = 1$, for some i we must have $T_i \cap H^{2,0}(X) = (0)$. But then $T_i \subset NS(X,\mathbb{Q})$, a contradiction.) Thus, if X and Y are algebraic surfaces with geometric genus 1, every non-zero element of $HKT(X,Y)$ is a cohomological isogeny (since every morphism of rational Hodge structures $T(X,\mathbb{Q}) \to T(Y,\mathbb{Q})$ is either the zero map or an isomorphism).

In particular, the Hodge conjecture for $X \times Y$ can be phrased in the following way: is every cohomological isogeny between X and Y a $\mathbb{Q}$-linear combination of cohomology classes of isogenies? Before addressing this question directly, we will investigate the Hodge theory of the situation a bit further.

<u>Theorem 1</u>. Let X be an algebraic surface with geometric genus 1 such that the minimal model of X is neither a K3 surface nor a logarithmic transform of an elliptic K3 surface. Then there exists a unique (algebraic) K3 surface Y with the property that there exists a strict integral cohomological

isogeny between X and Y. We call Y the K3 surface associated to X.

I want to discuss the proof of this theorem (which was originally given in [6]) in some detail here, because it relies on the Nikulin embedding theorem, which is an important new tool in the study of algebraic surfaces over the complex numbers. To formulate the Nikulin embedding theorem, we need some definitions.

A lattice is a free $\mathbb{Z}$-module L of finite rank, together with a nondegenerate $\mathbb{Z}$-valued symmetric bilinear form b on L. A lattice is even if $b(x,x) \in 2\mathbb{Z}$ for all $x \in L$.

Given a lattice L, the form b induces a natural inclusion $L \hookrightarrow L^*$ whose cokernel L^*/L is a finite abelian group (this is equivalent to the nondegeneracy of b), called the discriminant-group of L. We let $\ell(L^*/L)$ denote the minimum number of generators of the finite abelian group L^*/L. A lattice L is unimodular if, equivalently, $\ell(L^*/L) = 0$, L^*/L is trivial, or $L \simeq L^*$.

Every lattice L has a signature, which is a pair (s,t) of natural numbers; s (resp. t) is the dimension of the positive (resp. negative) eigenspace of the induced bilinear form on $L \otimes \mathbb{R}$. $s+t = rk(L)$ is the rank of L since b is nondegenerate.

An embedding of lattices $M \hookrightarrow L$ is primitive if L/M is torsion-free.

We can now state the Nikulin embedding theorem.

Theorem (Nikulin [12]).

(1) If L is a unimodular lattice, $\phi\colon M \hookrightarrow L$ is a primitive embedding, and $N = \phi(M)^{\perp}$, then $N^*/N \simeq M^*/M$ and

$$\ell(M^*/M) = \ell(N^*/N) \leq \mathrm{rk}(N) = \mathrm{rk}(L) - \mathrm{rk}(M).$$

(2) Let L be an even unimodular lattice with signature (s,t), and let M be an even lattice with signature (s',t') such that $s-s' > 0$ and $t-t' > 0$.

(i) If $\ell(M^*/M) \leq \mathrm{rk}(L) - \mathrm{rk}(M) - 1$, or if $\mathrm{rk}(M) \leq \frac{1}{2}\,\mathrm{rk}(L)$, then there exists a primitive embedding $\phi\colon M \hookrightarrow L$.

(ii) If $\ell(M^*/M) \leq \mathrm{rk}(L) - \mathrm{rk}(M) - 2$, then the primitive embedding ϕ is unique.

If Y is a K3 surface, $H^2(Y,\mathbb{Z})$ is an even unimodular lattice of signature (3,19) which does not depend on Y. We let Λ denote this lattice.

Proposition. Let X be an algebraic surface as in theorem 1. Then there is a unique primitive embedding $\phi\colon T(X,\mathbb{Z}) \hookrightarrow \Lambda$.

Proof: Since $T(X,\mathbb{Z})$ is a birational invariant, we may assume without loss of generality that X is minimal.

We first claim that there exists a unimodular lattice L with $\mathrm{rk}(L) \leq 20$ and a primitive embedding $\psi\colon T(X,\mathbb{Z}) \hookrightarrow L$. To show this, we use Noether's formula

$$c_1^2(X) + c_2(X) = 12\,\chi(\mathcal{O}_X).$$

Using the fact that the geometric genus $p_g = 1$ and that $c_2(X)$ is the topological Euler characteristic of X, we may rewrite this formula as

$$b_2(X) = 22 - 8q(X) - c_1^2(X)$$

where b_2 is the second Betti number and q is the irregularity.

If $c_1^2(X) \geq 2$ or $q(X) \geq 1$, we take $L = H^2(X,\mathbb{Z})$, which has rank $b_2(X) \leq 20$, and let ψ be the natural inclusion $T(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})$.

If $c_1^2(X) = 1$ and $q(X) = 0$ then $b_2(X) = 21$; let $L = (c_1(X))^{\perp} \subset H^2(X,\mathbb{Z})$. This is unimodular since $c_1^2(X) = 1$; moreover, since $c_1(X)$ is perpendicular to $T(X,\mathbb{Z})$, we have a natural embedding ψ: $T(X,\mathbb{Z}) \hookrightarrow L$.

Finally, if $c_1^2(X) \leq 0$ and $q(X) = 0$, since $p_g(X) = 1$ the Enriques-Kodaira classification of minimal surfaces guarantees that X is either a K3 surface or a logarithmic transform of an elliptic K3 surface, contrary to our assumptions.

Now by part (1) of the Nikulin embedding theorem,

$$\begin{aligned} \ell(T(X,\mathbb{Z})^*/T(X,\mathbb{Z})) &\leq \mathrm{rk}(L) - \mathrm{rk}(T(X,\mathbb{Z})) \\ &\leq 20 - \mathrm{rk}(T(X,\mathbb{Z})) \\ &= \mathrm{rk}(\Lambda) - \mathrm{rk}(T(X,\mathbb{Z})) - 2. \end{aligned}$$

In addition, the signature of $T(X,\mathbb{Z})$ is $(2, \mathrm{rk}(T(X,\mathbb{Z})) - 2)$ while that of Λ is $(3,19)$; since $\mathrm{rk}(T(X,\mathbb{Z})) \leq 20$, we see that $\mathrm{rk}(T(X,\mathbb{Z})) - 2 < 19$. Finally, $T(X,\mathbb{Z})$ is even by the Wu formula, which says that $\frac{1}{2}(\alpha.\alpha + \alpha.c_1(X))$ is an integer for any $\alpha \in H^2(X,\mathbb{Z})$; for $\alpha \in T(X,\mathbb{Z})$ of course, $\alpha.c_1(X) = 0$.

Thus, we may apply part (2) of the Nikulin embedding theorem, which guarantees the existence of a unique primitive embedding $\phi\colon T(X,\mathbb{Z}) \hookrightarrow \Lambda$. Q.E.D.

Proof of theorem 1: By the proposition, there is a primitive embedding $\phi\colon T(X,\mathbb{Z}) \hookrightarrow \Lambda$. Put a Hodge structure on Λ by using the Hodge structure of X on $\phi(T(X,\mathbb{Z}))$, and requiring that $\phi(T(X,\mathbb{Z}))^{\perp} \subset \Lambda^{1,1}$. By the surjectivity of the period map for K3 surfaces [3], [21], there exists a K3 surface Y with that Hodge structure. ϕ thus induces a strict integral cohomological isogeny between X and Y, and gives a morphism of Hodge structures $\phi\colon T(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$.

Suppose that Y' is another K3 surface such that there exists a strict integral cohomological isogeny between X and Y'. Then there is an isomorphism of Hodge structures $T(X,\mathbb{Z}) \simeq T(Y',\mathbb{Z})$, which gives an embedding $\phi'\colon T(X,\mathbb{Z}) \hookrightarrow H^2(Y',\mathbb{Z})$. By the uniqueness of the primitive embedding ϕ, ϕ' is isomorphic to ϕ; this means that there is an isometry $\psi\colon H^2(Y,\mathbb{Z}) \xrightarrow{\sim} H^2(Y',\mathbb{Z})$ such that $\psi \circ \phi = \phi'$. But then ψ is an isomorphism of Hodge structures; by the global Torelli theorem for K3 surfaces [14], Y' is isomorphic to Y. Q.E.D.

We should point out that the conclusion of theorem 1 does not hold if X is itself an algebraic K3 surface. Indeed, there exist non-isomorphic algebraic K3 surfaces X_1 and X_2 such that there is a strict integral cohomological isogeny between X_1 and X_2. This happens any time the lattice

$T(X_1,\mathbb{Z})$ admits several non-isomorphic primitive embeddings into Λ. (For then by the surjectivity of the period map, a K3 surface exists whose periods correspond to each of the embeddings.) However, if $\rho(X) \geq 12$ then this phenomenon cannot occur: in that case, part (2) of the Nikulin embedding theorem guarantees that the embedding $T(X,\mathbb{Z}) \hookrightarrow \Lambda$ is unique (cf. [5; cor. 2.10]).

Theorem 1 allows us to divide our search for isogenies into 2 parts: we search for strict integral isogenies which realize the cohomological isogenies given in theorem 1, and also for arbitrary isogenies between algebraic K3 surfaces. In fact, given a cohomological isogeny $T(X_1,\mathbb{Q}) \xrightarrow{\sim} T(X_2,\mathbb{Q})$ between two algebraic surfaces with geometric genus 1, if we let Y_1 and Y_2 be the associated K3 surfaces, there is a diagram

$$\begin{array}{ccc} T(X_1,\mathbb{Q}) & \xrightarrow{\sim} & T(X_2,\mathbb{Q}) \\ \downarrow\wr & & \downarrow\wr \\ T(Y_1,\mathbb{Q}) & & T(Y_2,\mathbb{Q}) \end{array} .$$

The vertical maps are strict integral cohomological isogenies, and the composite of the three maps is an arbitrary cohomological isogeny between K3 surfaces.

For a self-product of an algebraic K3 surface, there is a bit of extra information about the Hodge theory. Let X be an algebraic K3 surface, and let End(X) be the algebra of endomorphisms of $T(X,\mathbb{Q})$ which preserve the rational Hodge structure. The non-zero elements of End(X) are exactly the cohomological isogenies between X and X.

Theorem (Zarhin [23]). Let X be an algebraic K3 surface, let $d = \dim_{\mathbb{Q}} \mathrm{End}(X)$, and $t = \dim_{\mathbb{Q}} T(X,\mathbb{Q})$. Then

(1) End(X) is a subfield of $\mathbb{C}$ via the natural homomorphism

$$\varepsilon\colon \mathrm{End}(X) \hookrightarrow \mathrm{End}_{\mathbb{C}} H^{2,0}(X) \approx \mathbb{C}.$$

(2) $\alpha \in \mathrm{End}(X)$ is a strict cohomological isogeny if and only if $\varepsilon(\alpha)\ \overline{\varepsilon(\alpha)} = 1$.

(3) $d|t$, and either

(a) End(X) is a totally real number field, and $(t/d) \geq 3$, or

(b) End(X) is a totally imaginary quadratic extension of a totally real number field, and d and t are even integers.

Corollary. If X is an algebraic K3 surface with $\rho(X) = 3, 5, 9, 11, 15, 17$, or 19, then $\mathrm{Hdg}(X\times X)$ is generated by $\mathrm{Div}(X\times X)$ and the cohomology class of the diagonal.

Proof: The set of Picard numbers given is exactly the set such that $t = 22 - \rho(X)$ is an odd prime; this implies that End(X) falls in case (a). Since t is prime and $(t/d) \geq 3$, we see that $d = 1$ and $\mathrm{End}(X) = \mathbb{Q}$. In particular, $\dim \mathrm{HKT}(X,X) = 1$. But then the class of the diagonal must span $\mathrm{HKT}(X,X)$. Q.E.D.

We will now briefly discuss several constructions for isogenies between algebraic surfaces with geometric genus 1. We begin with the case of a product of two K3 surfaces.

Theorem (Mukai [8]). Let X and Y be algebraic K3 surfaces, and let $\phi: T(X,\mathbb{Q}) \xrightarrow{\sim} T(Y,\mathbb{Q})$ be a strict cohomological isogeny. Suppose that the lattice $T = T(X,\mathbb{Z}) \cap \phi^{-1}(T(Y,\mathbb{Z}))$ can be primitively embedded in the K3 lattice Λ. Then there exists an isogeny between X and Y inducing some multiple of ϕ.

Idea of the proof: Since T can be embedded in Λ, by the surjectivity of the period map there exists an algebraic K3 surface Z and an isomorphism of Hodge structures $T(Z,\mathbb{Z}) \xrightarrow{\sim} T$. It thus suffices to consider the case of an embedding $\psi: T(Z,\mathbb{Z}) \hookrightarrow T(X,\mathbb{Z})$ with finite cokernel. Such an embedding can be factored into a sequence $T(Z,\mathbb{Z}) = T_1 \subset T_2 \subset \dots \subset T_n = T(X,\mathbb{Z})$ in such a way that T_{i+1}/T_i is a finite cyclic group for each i. The theorem above now follows by induction from the following theorem.

Theorem (Mukai [8]). Let Z be an algebraic K3 surface, and let $\xi: T(Z,\mathbb{Z}) \hookrightarrow T$ be an embedding of lattices with finite cyclic cokernel; give T the induced Hodge structure. Then there is a compact component W of the moduli space of stable sheaves on Z which is an algebraic K3 surface, and an isomorphism of Hodge structures $\zeta: T \xrightarrow{\sim} T(W,\mathbb{Z})$ preserving the quadratic forms. Moreover, there is a "quasi-universal bundle" on Z×W whose Chern classes give a cycle on $Z\times W$ realizing some multiple of the strict cohomological isogeny $\zeta\circ\xi\colon T(Z,\mathbb{Z})\to T(W,\mathbb{Z})$.

From the first theorem of Mukai cited above, we get a corollary.

Corollary. Let X and Y be algebraic K3 surfaces.

(1) If $\phi\colon T(X,\mathbb{Q})\xrightarrow{\sim} T(Y,\mathbb{Q})$ is a strict integral cohomological isogeny, then there is an algebraic cycle realizing some multiple of ϕ.

(2) If $\rho(X)\geq 11$, then every strict cohomological isogeny $T(X,\mathbb{Q})\xrightarrow{\sim} T(Y,\mathbb{Q})$ is realized by a $\mathbb{Q}$-linear combination of algebraic cycles.

(3) If $\rho(X)\geq 11$ and End(X) is a cyclotomic field, then the Hodge conjecture holds for $X\times X$.

Proof: (1) is immediate, since $T(X,\mathbb{Z})\cap\phi^{-1}(T(Y,\mathbb{Z}))=T(X,\mathbb{Z})$ can be primitively embedded in Λ.

(2) follows from part (2)(i) of the Nikulin embedding theorem: if $\rho(X)\geq 11$, then $\mathrm{rk}(T)=\mathrm{rk}(T(X,\mathbb{Z}))\leq 11=\frac{1}{2}\,\mathrm{rk}(\Lambda)$, so that a primitive embedding of T into Λ exists.

To prove (3), we only need to show that $HKT(X,X)=\mathrm{End}(X)$ is generated over $\mathbb{Q}$ by the classes of algebraic cycles. By part (2) and Zarhin's theorem, the subspace spanned by the classes of algebraic cycles contains all $\alpha\in\mathrm{End}(X)$ such that $\varepsilon(\alpha)\,\overline{\varepsilon(\alpha)}=1$. But since End(X) is a cyclotomic field, it is generated over $\mathbb{Q}$ by such α's. Q.E.D.

We now turn to cycles on products $X \times Y$, where X is an algebraic surface with geometric genus 1, and Y is the K3 surface associated to X. In the case in which X is an abelian surface, we proved in [5] the following generalization of a theorem of Shioda and Inose [18].

Theorem 2. Let A be an abelian surface, and let Y be the K3 surface associated to A. Then there exists a strict integral isogeny between A and Y.

We will sketch the proof, which again uses the Nikulin embedding theorem; details can be found in [5]. If we let X be the minimal resolution of the Kummer surface $A/(\underline{+}1)$, then there is a rational map of degree two $f: A \dashrightarrow X$. The orthogonal complement of the image of the cohomology of A in $H^2(X,\mathbb{Z})$ gives a primitive sublattice $K \subset NS(X)$. Using the Nikulin embedding theorem, we show that the existence of a primitive embedding of K into $NS(X)$ implies the existence of a primitive embedding of another lattice $N \oplus E_8(-1)$ into $NS(X)$. An embedding of this second lattice is exactly what is needed to guarantee that there is a rational double cover $g: Y' \dashrightarrow X$ which (combined with f) induces a strict integral cohomological isogeny between A and Y'. The closure in $A \times Y'$ of the set

$$\{ (x,y) \in A \times Y' \mid f(x) = g(y) \}$$

now gives a strict integral isogeny between A and Y'; moreover, by the global Torelli theorem, Y is isomorphic to Y'.

Corollary.

(1) If A is an abelian surface and Y is the K3 surface associated to A, then the Hodge conjecture holds for $Y \times Y$.

(2) If Y is an algebraic K3 surface with $\rho(Y) = 19$ or 20, then the Hodge conjecture holds for $Y \times Y$.

Proof: By the theorem of Mumford-Tanke'ev-Ribet-Murty mentioned earlier, $\mathrm{Div}(A \times A) = \mathrm{Hdg}(A \times A)$. Using the isogeny constructed in theorem 2, we may transfer the algebraic cycles which span HKT(A,A) to algebraic cycles which span HKT(Y,Y), proving (1).

(2) follows from (1), combined with the fact that every algebraic K3 surface with Picard number 19 or 20 is the K3 surface associated to some abelian surface A. (This last fact is also proved using the Nikulin embedding theorem: cf. [5; cor. 6.4]). Q.E.D.

I have recently been studying cycles on products $X \times Y$, where X is one of the surfaces of general type with geometric genus 1 introduced by Todorov [22] (which I call Todorov surfaces), and Y is the K3 surface associated to X. For these surfaces, similar techniques produce a strict integral isogeny between X and Y; details will appear in [7]. However, Todorov surfaces have deformations which are not themselves Todorov surfaces, and I do not yet know how to construct the analogous cycles for these deformations, or whether those cycles exist.

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