

## Cremona Transformations and Degrees of Period Maps for $K3$ Surfaces with Ordinary Double Points

David R. Morrison<sup>1</sup> and Masa-Hiko Saito<sup>2</sup>

Let  $X$  be a  $K3$  surface with  $n$  ordinary double points on which an ample line bundle  $\mathcal{L}$  has been fixed. If  $\mu: S \rightarrow X$  is the minimal desingularization, the orthogonal complement in  $H^2(S, \mathbb{Z})$  of  $\mu^*\mathcal{L}$  and the rational curves  $\mu^{-1}(P)$  (for  $P \in \text{Sing } X$ ) carries a Hodge structure with  $h^{2,0} = h^{0,2} = 1$  and  $h^{1,1} = 19 - n$ . The *period map* for these surfaces is the natural map from the moduli space to the classifying space for such Hodge structures; this generalizes the classical period maps for polarized  $K3$  surfaces [24].

In contrast to the classical case, these period maps, while always étale, tend to have degree greater than one. In this paper, we study this phenomenon for two particular kinds of  $K3$  surfaces with ordinary double points: the “ $K3$  surfaces of Cremona type” in which the degree of  $\mathcal{L}$  is 2 and the branch locus of the induced map to  $\mathbb{P}^2$  is irreducible, and the “ $K3$  surfaces of Todorov type” whose study was begun in [20]; these latter surfaces arise as quotients of certain surfaces of general type constructed by Todorov [26]. (We omit one of the families of  $K3$  surfaces of Todorov type here, as to consider it would take us too far afield.)

Our main results are a computation of the degrees of the period maps (Corollaries (5.2) and (5.6)), a demonstration that for our families, two  $K3$  surfaces with the same periods are birationally (but not always biregularly) isomorphic (Theorem (6.1)), and finally a consideration of the geometric consequences of these birational isomorphisms when the degree of  $\mathcal{L}$  is small (Theorems (7.1), (7.3), and (8.5)). The geometric consequences we find involve the behavior of sets of points in  $\mathbb{P}^2$  or  $\mathbb{P}^3$  under the Cremona group of birational automorphisms of  $\mathbb{P}^2$  or  $\mathbb{P}^3$  (hence the name “Cremona type”), and we obtain modern proofs of some classical results of Coble [3], [4], [5] on this topic. We should mention that Coble’s work has recently been studied from a different point of view by Cossec and Dolgachev [8], [6], whose results we use in our interpretation.

Received May 14, 1986.

<sup>1</sup> Research supported by the Japan Society for the Promotion of Science and the National Science Foundation.

<sup>2</sup> Research partially supported by the Educational Project for Japanese Mathematical Scientists.

The plan of the paper is as follows: we begin with some algebraic preliminaries on finite quadratic forms, their orthogonal groups, and the "Cremona and Todorov lattices" in the first three sections. Section 4 is devoted to the construction of moduli spaces for  $K3$  surfaces of Cremona type. (The analogous construction for  $K3$  surfaces of Todorov type was given in [20]). In Section 5 we compute the degrees of the period maps; we then find a link between some of these period maps and birational geometry in  $\mathbf{P}^2$  and  $\mathbf{P}^3$  in the last three sections.

We would like to thank Igor Dolgachev and Rick Miranda for helpful discussions during the preparation of this paper.

### §1. Finite quadratic forms

A *finite quadratic form* is a pair  $(G, q)$  (sometimes denoted simply by  $G$ ) consisting of a finite abelian group  $G$  together with a map  $q: G \rightarrow \mathbf{Q}/2\mathbf{Z}$  which satisfies (1)  $q(nx) \equiv n^2q(x) \pmod{2\mathbf{Z}}$  for all  $n \in \mathbf{Z}$ ,  $x \in G$ , and (2) the map  $b: G \times G \rightarrow \mathbf{Q}/\mathbf{Z}$  defined by  $b(x, x') \equiv (q(x+x') - q(x) - q(x'))/2 \pmod{\mathbf{Z}}$  is symmetric and  $\mathbf{Z}$ -bilinear.  $b$  is called the *associated bilinear form* of  $(G, q)$ .

If  $(G, q)$  is a finite quadratic form and  $H$  is a subgroup of  $G$ , we define  $H^\perp = \{x \in G \mid b(x, u) \equiv 0 \pmod{\mathbf{Z}} \text{ for all } u \in H\}$ .  $H^\perp$  is called the *radical* of  $G$ , and denoted by  $\text{Rad}(G)$ ; the  *$q$ -radical* of  $G$  is the subgroup  $\text{Rad}_q(G) = \{x \in \text{Rad}(G) \mid q(x) \equiv 0 \pmod{2\mathbf{Z}}\}$ . The form  $(G, q)$  is *nondegenerate* if  $\text{Rad}(G) = 0$ , and *quasi-nondegenerate* if  $\text{Rad}_q(G) = 0$ .

Let  $G_i$  be a subgroup of  $G$  with induced quadratic form  $q_i = q|_{G_i}$  for  $i=1, 2$ . If  $G_2 = G_1^\perp$  and  $G = G_1 \oplus G_2$ , we say that  $(G, q)$  is the *orthogonal direct sum* of  $(G_1, q_1)$  and  $(G_2, q_2)$ , and denote this by writing  $(G, q) = (G_1, q_1) \oplus (G_2, q_2)$ .

**Lemma (1.1).** *Let  $(G, q)$  be a finite quadratic form, and let  $H$  be a subgroup of  $G$  such that  $q|_H$  is a nondegenerate form. Then  $(G, q)$  is the orthogonal direct sum of  $(H, q|_H)$  and  $(H^\perp, q|_{H^\perp})$ .*

*Proof.* First note that  $H \cap H^\perp = \text{Rad}(H)$ , so that  $H \cap H^\perp = 0$ . To show that  $H + H^\perp$  coincides with  $G$ , consider the homomorphism  $H \rightarrow \text{Hom}(H, \mathbf{Q}/\mathbf{Z})$  defined by the adjoint map of the bilinear form  $b$ . This is injective since  $H$  is nondegenerate; on the other hand, it is a homomorphism between two groups of the same order, so it must be an isomorphism.

Now given any  $x \in G$ , the map  $u \rightarrow b(u, x)$  defines an element of  $\text{Hom}(H, \mathbf{Q}/\mathbf{Z})$ . There must then be some  $y \in H$  such that the map  $u \rightarrow b(u, y)$  defines the same element of  $\text{Hom}(H, \mathbf{Q}/\mathbf{Z})$ . But this means that  $x - y \in H^\perp$ ; hence,  $G = H + H^\perp$ . Q.E.D.

We introduce some notation for quadratic forms on cyclic groups. If  $a$  and  $l$  are natural numbers with  $2 \mid al$  and  $(a, l) = 1$ , we let  $z_l^a$  denote the form  $(\mathbf{Z}/l\mathbf{Z}, q)$ , where  $q(x) = a/l \pmod{2\mathbf{Z}}$  for some generator  $x$  of  $\mathbf{Z}/l\mathbf{Z}$ . Following Brieskorn [1], when  $\varepsilon \in \{1, 3, 5, 7\}$  we let  $w_{2,k}^\varepsilon = z_{2^k}^\varepsilon$ , and when  $p$  is an odd prime and  $\varepsilon = \pm 1$  we let  $w_{p,k}^\varepsilon = z_{p^k}^\varepsilon$  for some even integer  $a$  with  $\left(\frac{a}{p}\right) = \varepsilon$  (where  $(-)$  denotes the Legendre symbol).

Let  $(G, q)$  be a nondegenerate quadratic form. The form  $(G, q)$  is *indecomposable* if  $(G, q)$  cannot be written as the orthogonal direct sum of two nontrivial forms. The cyclic forms  $w_{p,k}^\varepsilon$  provide examples of indecomposable forms. Other examples are given by  $u_k$  and  $v_k$  (again following Brieskorn's notation), which are forms on  $\mathbf{Z}/2^k\mathbf{Z} \times \mathbf{Z}/2^k\mathbf{Z}$ , where on a generating set  $x, y$  we have  $q(x) \equiv q(y) \equiv 0 \pmod{2\mathbf{Z}}$  and  $b(x, y) \equiv 2^{-k} \pmod{\mathbf{Z}}$  for  $u_k$ , and  $q(x) \equiv q(y) \equiv 2^{1-k} \pmod{2\mathbf{Z}}$  and  $b(x, y) \equiv 2^{-k} \pmod{\mathbf{Z}}$  for  $v_k$ .

The following proposition is well-known (cf. [27], [9], [22]).

**Proposition (1.2).** (1) *Let  $(G, q)$  be a nondegenerate quadratic form, let  $G = \bigoplus G_p$  be the decomposition of  $G$  into its  $p$ -Sylow subgroups, and let  $q_p = q|_{G_p}$ . Then  $(G, q) = \bigoplus (G_p, q_p)$  is an orthogonal direct sum decomposition.*

(2) *Every nondegenerate finite quadratic form is isomorphic to an orthogonal direct sum of indecomposable forms.*

(3) *A nondegenerate finite quadratic form is indecomposable if and only if it is isomorphic to one of the forms  $u_k, v_k$ , or  $w_{p,k}^\varepsilon$ ; in particular, these forms generate the semigroup  $\text{qu}(\mathbf{Z})$  of isomorphism classes of nondegenerate finite quadratic forms, where orthogonal direct sum is the semigroup operation.*

(4) *For an odd prime  $p$ ,  $(w_{p,k}^{-1})^{\otimes 2} \cong (w_{p,k}^1)^{\otimes 2}$ .*

(5) *The following relations hold among indecomposable forms on 2-groups. (We identify  $\{1, 3, 5, 7\}$  with  $(\mathbf{Z}/8\mathbf{Z})^\times$ .)*

$$(I) \quad w_{2,1}^5 \cong w_{2,1}^1 \quad \text{and} \quad w_{2,1}^3 \cong w_{2,1}^7.$$

$$(II) \quad w_{2,k}^{\varepsilon_1} \oplus w_{2,k}^{\varepsilon_2} \oplus w_{2,k}^{\varepsilon_3} \cong \begin{cases} u_k \oplus w_{2,k}^{s_2(\varepsilon)} & \text{if } s_2(\varepsilon) = 7 \\ v_k \oplus w_{2,k}^{s_2(\varepsilon)} & \text{if } s_2(\varepsilon) = 3 \end{cases}$$

where  $s_1(\varepsilon) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$  and  $s_2(\varepsilon) = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$ .

$$(III) \quad u_k^{\otimes 2} \cong v_k^{\otimes 2}.$$

**Remark.** There are more relations among indecomposable forms than those listed in (4) and (5) above; for a more complete account, see [10] or [17].

We now use the relations in (5) above to derive a nice description of quadratic forms on 2-groups.

**Proposition (1.3)** (cf. [17], [18]). *Let  $(G, q)$  be a nondegenerate finite quadratic form with  $p$ -Sylow decomposition  $(G, q) = \bigoplus (G_p, q_p)$ .*

(1) *If  $p$  is odd, then  $(G_p, q_p)$  has an orthogonal direct sum decomposition*

$$(G_p, q_p) \cong \bigoplus_{k \geq 1} ((w_{p,k}^1)^{\oplus a(p,k)} \oplus (w_{p,k}^{-1})^{\oplus b(p,k)})$$

with  $b(p, k) \leq 1$ .

(2)  *$(G_2, q_2)$  has an orthogonal direct sum decomposition*

$$(G_2, q_2) \cong \bigoplus_{k \geq 1} (u_k^{\oplus n(k)} \oplus v_k^{\oplus m(k)} \oplus w(k))$$

with the following properties:

- (a)  $m(k) \leq 1$ ,
- (b)  $\text{rank}(w(k)) \leq 2$ ,
- (c)  $w(k)$  is a sum of forms of type  $w_{2,k}^{\pm}$ ,
- (d)  $w(1)$  is a sum of forms of type  $w_{2,1}^{\pm}$  and  $w_{2,1}^{\mp}$ .

*Proof.* We decompose  $(G, q)$  as an orthogonal direct sum of indecomposable forms; (1) is a direct consequence of Proposition (1.2) (4). To prove (2), we apply the relations from Proposition (1.2) (5): first, for each  $k$ , by repeated applications of relation (II) we may reduce to the case in which there are at most 2 summands of type  $w_{2,k}^{\pm}$ . Next, a repeated application of relation (III) will reduce the number of summands of type  $v_k$  to at most 1. Finally, by applying relation (I) we may ensure that condition (d) holds. Q.E.D.

We say that a decomposition satisfying the conditions in Proposition (1.3) is in *normal form*. Such a “normal form” is not in general unique, but the first terms in the normal form decomposition are unique in the following sense:

**Proposition (1.4).** *Fix natural numbers  $n$  and  $m$ , and let*

$$(G, q) \cong u_1^{\oplus n} \oplus v_1^{\oplus m} \oplus (G', q')$$

*be a nondegenerate finite quadratic form. If  $H$  is any subgroup of  $G$  such that  $(H, q|_H) \cong u_1^{\oplus n} \oplus v_1^{\oplus m}$ , then  $G$  is the orthogonal direct sum of  $H$  and  $H^\perp$ , and  $(H^\perp, q|_{H^\perp}) \cong (G', q')$ .*

*Proof.* We use the signature invariants  $\sigma_r(G, q)$  of a nondegenerate finite quadratic form<sup>3</sup> introduced by Kawauchi and Kojima [10] (cf. also [17]). These invariants, which are defined for  $r \geq 1$ , take values in the

<sup>3</sup> In [10] and [17], these invariants are defined for bilinear rather than quadratic forms; we are implicitly using a technique of Wall [27; Theorem 5], which associates a bilinear form to each quadratic form in such a way as to embed  $\text{qu}(\mathbb{Z})$  in the semigroup of isomorphism classes of nondegenerate finite bilinear forms.

semigroup  $\bar{\mathbf{Z}}_8 = (\mathbf{Z}/8\mathbf{Z}) \cup \{\infty\}$  (in which addition mod 8 is extended by the rules  $i + \infty = \infty + i = \infty + \infty = \infty$ ) and  $\sigma_r: qu(\mathbf{Z}) \rightarrow \bar{\mathbf{Z}}_8$  is a semigroup homomorphism. The values of these invariants for the indecomposable forms are given in Table 1.

The fundamental theorem about the signature invariants is this: if  $G_i$  is a 2-group and  $(G_i, q_i)$  is a nondegenerate finite quadratic form for  $i=1, 2$  such that  $G_1 \cong G_2$  (as groups) and  $\sigma_r(G_1, q_1) = \sigma_r(G_2, q_2)$  for all  $r \geq 1$ , then  $(G_1, q_1) \cong (G_2, q_2)$ . Applied to the present situation, by using the Sylow decomposition, we may reduce to the case in which  $G$  is a 2-group. It is clear from Lemma (1.1) that  $(G, q) \cong (H, q|_H) \oplus (H^\perp, q|_{H^\perp})$  and that  $H^\perp \cong G'$  as groups. Now since  $\sigma_r(u_1^{\oplus n} \oplus v_1^{\oplus m}) = n\sigma_r(u_1) + m\sigma_r(v_1) \in \{0, 4\}$  always has an additive inverse in  $\bar{\mathbf{Z}}_8$  we see that

$$\begin{aligned} \sigma_r(H^\perp, q|_{H^\perp}) &= \sigma_r(G, q) - \sigma_r(H, q|_H) \\ &= \sigma_r(G, q) - \sigma_r(u_1^{\oplus n} \oplus v_1^{\oplus m}) \\ &= \sigma_r(G', q'). \end{aligned}$$

Hence,  $(H^\perp, q|_{H^\perp}) \cong (G', q')$ .

Q.E.D.

Table 1. Signature invariants of indecomposable forms

$(G, q)$	$p$ or $r$	$\sigma_r(G, q)$
$w_{p,k}^\varepsilon$	$p \neq 2$	$(-1)^{(r-1)(p^2-1)/8}(k^2(1-p) + 2k(1-\varepsilon))$
$w_{2,k}^\varepsilon$	$r \leq k$	$\{(-1)^{(s-1)/2}(1 + (-1)^{k+r}) + \varepsilon(1 - (-1)^{k+r})\}/2$
$w_{2,k}^\varepsilon$	$r = k + 1$	$\infty$
$w_{2,k}^\varepsilon$	$r \geq k + 2$	$0$
$u_k$	all $r$	$0$
$v_k$	$r \leq k$	$2(1 + (-1)^{k+r})$
$v_k$	$r \geq k + 1$	$0$

§ 2. Finite orthogonal groups

For a finite quadratic form  $(G, q)$ , we define the *orthogonal group of  $(G, q)$*  by  $O(G, q) = \{g \in \text{Aut}(G) \mid q(g(x)) = q(x) \text{ for all } x \in G\}$ . In this section we study the structure of  $O(G, q)$  (and in particular calculate its order) in several special cases. We begin with some examples.

**Examples (2.1).** (1) If  $(G, q) = \bigoplus (G_p, q_p)$  is the decomposition of  $G$  into its  $p$ -Sylow subgroups, then there is a natural isomorphism  $O(G, q) \cong \prod O(G_p, q_p)$ .

(2) Let  $a$  and  $l$  be natural numbers with  $2 \mid al$  and  $(a, l) = 1$ . Then

$$O(z_l^a) \cong \begin{cases} \{\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times \mid \alpha^2 \equiv 1 \pmod{l}\} & \text{if } l \text{ is odd} \\ \{\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times \mid \alpha^2 \equiv 1 \pmod{2l}\} & \text{if } l \text{ is even.} \end{cases}$$

In particular,

$$O(w_{p,k}^a) \cong \begin{cases} \{1\} & \text{if } p=2, k=1 \\ \mathbf{Z}/2\mathbf{Z} & \text{otherwise.} \end{cases}$$

(3)

$$O(w_{2,1}^\varepsilon \oplus w_{2,1}^\phi) \cong \begin{cases} \{1\} & \text{if } \varepsilon \equiv -\phi \pmod{4} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } \varepsilon \equiv \phi \pmod{4}. \end{cases}$$

*Proof.* Part (1) follows directly from Proposition (1.2) (1). To prove (2), let  $x$  be a generator of  $z_l^a$  such that  $q(x) = a/l$ . Any  $\gamma \in O(z_l^a)$  must be of the form  $\gamma(x) = \alpha x$  for some  $\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times$ . Since  $q(\alpha x) = a\alpha^2/l$ , such a  $\gamma$  belongs to  $O(z_l^a)$  if and only if  $a\alpha^2/l \equiv a/l \pmod{2\mathbf{Z}}$ , i.e.,  $a\alpha^2 \equiv a \pmod{2l}$ . If  $l$  is odd then  $a$  is even and this is equivalent to  $\alpha^2 \equiv 1 \pmod{l}$ ; if  $l$  is even then  $(a, 2l) = 1$  so that this is equivalent to  $\alpha^2 \equiv 1 \pmod{2l}$ .

For (3), let  $x$  and  $y$  be generators of  $w_{2,1}^\varepsilon$  and  $w_{2,1}^\phi$  respectively. The nonzero elements in the group are then  $x, y$  and  $x+y$  and on these elements the quadratic form takes values  $q(x) \equiv \varepsilon/2$ ,  $q(y) \equiv \phi/2$  and  $q(x+y) \equiv (\varepsilon + \phi)/2 \pmod{2\mathbf{Z}}$ . Since  $q(x+y) \equiv 0 \pmod{\mathbf{Z}}$ , the only possible nontrivial automorphism would be the one interchanging  $x$  and  $y$ ; this is an automorphism exactly when  $q(x) \equiv q(y) \pmod{2\mathbf{Z}}$ , i.e., when  $\varepsilon \equiv \phi \pmod{4}$ .

Q.E.D.

A finite quadratic form  $(G, q)$  is *special* if every  $x \in G$  with  $2x = 0$  satisfies  $q(x) \equiv 0 \pmod{\mathbf{Z}}$ . If  $(G, q)$  is a special quadratic form on a 2-elementary group, we may regard  $(G, q)$  as a quadratic form over the field with two elements  $F_2$  by giving  $G$  its natural  $F_2$ -vector space structure and identifying  $\mathbf{Z}/2\mathbf{Z}$  with  $F_2$ . The orthogonal group of  $(G, q)$  is then related to the classical orthogonal and symplectic groups over  $F_2$ .

**Proposition (2.2).** Let  $(G, q)$  be a special quadratic form on a 2-elementary group of rank  $k$ .

(1) If  $(G, q)$  is nondegenerate, then  $k = 2s$  is even and  $(G, q) \cong u_1^{\oplus s} \oplus v_1^{\oplus s}$  with  $m \leq 1$ . The orthogonal groups in these cases have been studied classically:  $O(u_1^{\oplus n})$  is usually called the even orthogonal group  $O^+(2n, 2)$ , and  $O(u_1^{\oplus n} \oplus v_1)$  is the odd orthogonal group  $O^-(2n+2, 2)$ .

(2) If  $(G, q)$  is degenerate and quasi-nondegenerate, then  $k=2n+1$  is odd and  $O(G, q) \cong Sp(2n, 2)$ , the symplectic group over the field  $F_2$ .

(3) In general let  $\bar{G} = G/\text{Rad}_q(G)$ , let  $r = \text{rank}(\text{Rad}_q(G))$ , and let  $s = k - r$ . Then  $\bar{G}$  inherits a quasi-nondegenerate special quadratic form  $\bar{q}$  from  $G$ , and there is an exact sequence

$$1 \longrightarrow T(r, s) \longrightarrow O(G, q) \longrightarrow O(\bar{G}, \bar{q}) \longrightarrow 1$$

where  $T(r, s) = \left\{ \begin{pmatrix} A & 0 \\ B & I \end{pmatrix} \in \text{GL}(r+s, F_2) \right\}$ ,  $I$  being the  $s \times s$  identity matrix.

*Proof.* (1) Let  $(G, q) \cong u_1^{\oplus n} \oplus v_1^{\oplus m} \oplus w(1)$  be a normal form decomposition. Since  $(G, q)$  is special,  $w(1)$  must be zero;  $G$  then has rank  $2n + 2m$ . For the identification of the orthogonal groups, see for example [25; pp. 380–382].

(2) If  $x \in \text{Rad}(G)$  and  $x \neq 0$  then  $q(x) \equiv 1 \pmod{2\mathbb{Z}}$ . Since  $q$  restricts to a linear function on  $\text{Rad}(G)$ , this implies  $\text{Rad}(G) \equiv \mathbb{Z}/2\mathbb{Z}$ ; we let  $x_0$  be the non-trivial element.

Let  $\tilde{G} = G/\text{Rad}(G)$  and define a bilinear form  $\tilde{b}: \tilde{G} \times \tilde{G} \rightarrow \mathbb{Z}/2\mathbb{Z}$  by  $\tilde{b}(\tilde{x}, \tilde{y}) \equiv 2b(x, y) \pmod{2\mathbb{Z}}$  for  $x$  and  $y$  in  $G$  mapping to  $\tilde{x}$  and  $\tilde{y}$  respectively.  $\tilde{b}$  is clearly nondegenerate; moreover  $\tilde{b}(\tilde{x}, \tilde{x}) \equiv 2b(x, x) \equiv 2q(x) \equiv 0 \pmod{2\mathbb{Z}}$  so that  $\tilde{b}$  is an alternating form. In particular,  $\tilde{G}$  has even rank  $2n$  so that  $k = 2n + 1$ . Moreover, there is a natural map  $f: O(G, q) \rightarrow O(\tilde{G}, \tilde{b}) = Sp(2n, 2)$ .

Let  $x'_1, \dots, x'_{2n}$  be elements of  $G$  mapping to a basis  $\tilde{x}_1, \dots, \tilde{x}_{2n}$  of  $\tilde{G}$ , and let  $x_i = x'_i + q(x'_i)x_0$  so that  $q(x_i) \equiv 0 \pmod{2\mathbb{Z}}$ . Then

$$(*) \quad q\left(\sum a_i x_i\right) \equiv \sum a_i a_j \tilde{b}(\tilde{x}_i, \tilde{x}_j).$$

Now given  $\sigma \in O(\tilde{G}, \tilde{b})$ , let  $\sigma(\tilde{x}_i) = \sum s_{ij} \tilde{x}_j$  and define  $g(\sigma): G \rightarrow G$  by  $g(\sigma)(x_0) = x_0$ ,  $g(\sigma)(x_i) = \sum s_{ij} x_j$ . By (\*), since  $\sigma$  preserves  $\tilde{b}$ ,  $g(\sigma)$  preserves  $q$  so that  $g$  gives a homomorphism  $g: O(\tilde{G}, \tilde{b}) \rightarrow O(G, q)$ . Since  $f \circ g$  is the identity,  $O(G, q)$  is isomorphic to  $O(\tilde{G}, \tilde{b}) = Sp(2n, 2)$ .

(3) Let  $f: O(G, q) \rightarrow O(\bar{G}, \bar{q})$  be the natural homomorphism. Choose a subgroup  $H$  of  $G$  such that  $G \cong \text{Rad}_q(G) \oplus H$  as groups. (This is possible since  $G$  is 2-elementary.) Then there is a natural isomorphism  $p: (H, q|_H) \xrightarrow{\sim} (\bar{G}, \bar{q})$ ; in particular, for any  $\tau \in O(\bar{G}, \bar{q})$  we have  $1 \oplus p^* \tau \in O(G, q)$  and  $f(1 \oplus p^* \tau) = \tau$  which shows that  $f$  is surjective.

If  $\sigma \in \text{Ker } f$  then  $\sigma(y) - y \in \text{Rad}_q(G)$  for any  $y \in H$ ; moreover  $\sigma(\text{Rad}_q(G)) = \text{Rad}_q(G)$ . This shows that  $\text{Ker } f \subset T(r, s)$ ; conversely, if  $\sigma \in T(r, s)$ ,  $x \in \text{Rad}_q(G)$  and  $y \in H$  then

$$q(\sigma(x+y)) \equiv q((\sigma(x) + \sigma(y) - y) + y) \equiv q(y) \equiv q(x+y)$$

since  $x, \sigma(x)$  and  $\sigma(y) - y$  all lie in  $\text{Rad}_q(G)$ . Thus,  $T(r, s)$  is contained in  $O(G, q)$ , and it clearly lies in the kernel of  $f$ . Q.E.D.

**Proposition (2.3).** *Let  $(G, q) = u_1^{\oplus n} \oplus v_1^{\oplus m} \oplus (G', q')$  be a nondegenerate quadratic form, and define  $K = \{x \in G \mid 2x = 0 \text{ and } q(x) \in \mathbf{Z}/2\mathbf{Z}\}$ . Then  $K$  is a subgroup of  $G$ , and  $(K, q|_K)$  is a special 2-elementary form. If  $K \cap G'$  has order less than or equal to 2, then there is an exact sequence*

$$1 \longrightarrow O(G', q') \longrightarrow O(G, q) \longrightarrow O(K, q|_K) \longrightarrow 1.$$

*Proof.* If  $x, y \in K$  then  $2b(x, y) \equiv b(2x, y) \equiv 0 \pmod{\mathbf{Z}}$ . Hence,  $q(x + y) \equiv q(x) + q(y) + 2b(x, y) \equiv 0 \pmod{\mathbf{Z}}$  so that  $K$  is indeed a subgroup of  $G$ .

Suppose that  $K \cap G'$  has order at most 2. Any  $\rho \in O(G, q)$  must preserve  $K$ , so there is a natural restriction homomorphism  $r: O(G, q) \rightarrow O(K, q|_K)$ . Since  $u_1^{\oplus n} \oplus v_1^{\oplus m} \subset K$ , any  $\rho \in O(G, q)$  which acts trivially on  $K$  must have the form  $\rho = 1 \oplus \nu$  with  $\nu \in O(G', q')$ . Moreover, any such element acts trivially on  $K$ : the induced action on  $K \cap G'$  is trivial since  $K \cap G'$  has order at most 2 (and hence has no non-trivial automorphisms). We conclude that the kernel of  $r$  is  $O(G', q')$ .

It remains to show that  $r$  is surjective. Let  $\sigma \in O(K, q|_K)$  and let  $H = \sigma(u_1^{\oplus n} \oplus v_1^{\oplus m})$ . By Proposition (1.4),  $(G, q) = (H, q|_H) \oplus (H^\perp, q|_{H^\perp})$  and there is an isomorphism  $\tau: (G', q') \rightarrow (H, q|_H)$ . Let  $\rho = (\sigma|_{u_1^{\oplus n} \oplus v_1^{\oplus m}}) \oplus \tau$ ; identifying  $G$  with  $H \oplus H^\perp$ , we may regard  $\rho$  as an element of  $O(G, q)$ . But then  $\rho|_K \circ \sigma^{-1}$  acts trivially on  $u_1^{\oplus n} \oplus v_1^{\oplus m}$ ; it thus induces an automorphism of  $K \cap G'$  which must be trivial as well. Hence,  $\rho|_K \circ \sigma^{-1} = 1_K$  which implies that  $\sigma$  is in the image of  $r$ . Q.E.D.

**Corollary (2.4).** *Let  $(G, q) = u_1^{\oplus n} \oplus v_1^{\oplus m} \oplus (G', q')$  be a nondegenerate quadratic form.*

(1) *If  $(G', q')$  is trivial, or  $(G', q') \cong w_{2,1}^2$ , then*

$$O(G, q) \cong O(u_1^{\oplus n} \oplus v_1^{\oplus m}).$$

(2) *If  $(G', q') \cong w_{2,2}^2$  or  $(G', q') \cong (w_{2,1}^2)^{\oplus 2}$ , then there is an exact sequence*

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow O(G, q) \longrightarrow Sp(2n + 2m, 2) \longrightarrow 1.$$

(3) *If  $(G', q') \cong w_{2,1}^1 \oplus w_{2,1}^1$ , then*

$$O(G, q) \cong (\mathbf{Z}/2\mathbf{Z})^{2n+2m} \times O(u_1^{\oplus n} \oplus v_1^{\oplus m}).$$

(4) *If  $(G', q') \cong w_{2,k}^2$  with  $k \geq 3$ , then there is an exact sequence*

$$1 \longrightarrow (\mathbf{Z}/2\mathbf{Z}) \longrightarrow O(G, q) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^{2n+2m} \times O(u_1^{\oplus n} \oplus v_1^{\oplus m}) \longrightarrow 1.$$

*Proof.* Let  $K$  be as in Proposition (2.3); we will show that in each case,  $K \cap G'$  has order at most 2, and we will compute  $O(G', q')$  and  $O(K, q|_K)$ . The statement is obvious when  $(G', q')$  is trivial.

Suppose first that  $(G', q') \cong w_{2,k}^\varepsilon$  and let  $x$  be a generator of  $G'$  so that  $q(2^{k-1}x) = 2^{k-2}\varepsilon$ . If  $k=1$  then  $K \cap G'$  is trivial and  $(K, q|_K) = u_1^{\oplus m} \oplus v_1^{\oplus m}$  is nondegenerate. By Example (2.1) (2),  $O(G', q')$  is trivial, so the stated isomorphism follows from Proposition (2.3).

If  $k=2$  then  $K$  is degenerate and quasi-nondegenerate with  $\text{Rad}(K) = K \cap G'$  generated by  $2x$ . The exact sequence follows from Example (2.1) (2) and Proposition (2.2) (2).

If  $k \geq 3$  then  $\text{Rad}_q(K) = K \cap G'$  is generated by  $2^{k-1}x$ , and the induced form on  $K/\text{Rad}_q(K)$  is isomorphic to  $u_1^{\oplus m} \oplus v_1^{\oplus m}$ . The exact sequence follows from Example (2.1) (2) and Proposition (2.2) (3).

Suppose now that  $(G', q') \cong w_{2,1}^\varepsilon \oplus w_{2,1}^\phi$ , and let  $x$  and  $y$  generate  $w_{2,1}^\varepsilon$  and  $w_{2,1}^\phi$  respectively. Then  $K \cap G'$  is generated by  $x + y$ , and  $q(x + y) = (\varepsilon + \phi)/2$ . If  $\varepsilon \equiv \phi \pmod{4}$  then  $K$  is degenerate and quasi-nondegenerate with  $\text{Rad}(K) = K \cap G'$ . On the other hand, if  $\varepsilon \equiv -\phi \pmod{4}$  then  $\text{Rad}_q(K) = K \cap G'$  and the induced form on  $K/\text{Rad}_q(K)$  is isomorphic to  $u_1^{\oplus m} \oplus v_1^{\oplus m}$ . In both cases, the structure of  $O(G, q)$  now follows from Example (2.1) (3) and Proposition (2.2). Q.E.D.

To finish the computation of the order of  $O(G, q)$  in the cases covered by Corollary (2.4), we recall (from [25; pp. 382, 392], for example) the orders of  $O^+(2n, 2)$ ,  $O^-(2n+2, 2)$ , and  $Sp(2n, 2)$ :

**Lemma (2.5).**

$$(1) \quad |O(u_1^{\oplus n})| = |O^+(2n, 2)| = 2 \cdot 2^{n(n-1)} \cdot (2^n - 1) \cdot \prod_{i=1}^{n-1} (2^{2i} - 1).$$

$$(2) \quad |O(u_1^{\oplus n} \oplus v_1)| = |O^-(2n+2, 2)| = 2 \cdot 2^{n(n+1)} \cdot (2^{n+1} + 1) \cdot \prod_{i=1}^n (2^{2i} - 1).$$

$$(3) \quad |Sp(2n, 2)| = 2^{n^2} \cdot \prod_{i=1}^n (2^{2i} - 1).$$

Table 2 shows the orders of these groups for low values of  $n$ . These can be combined with Corollary (2.4) to calculate  $|O(G, q)|$  in a number of cases; we will carry out the calculations in the cases of interest to us in the next section.

Table 2. Orders of certain finite orthogonal and symplectic groups

$k$	$ O^+(2k, 2) $	$ O^-(2k, 2) $	$ Sp(2k, 2) $
1	2	2 · 3	2 · 3
2	$2^3 · 3^2$	$2^3 · 3 · 5$	$2^4 · 3^2 · 5$
3	$2^7 · 3^2 · 5 · 7$	$2^7 · 3^4 · 5$	$2^9 · 3^4 · 5 · 7$
4	$2^{13} · 3^5 · 5^2 · 7$	$2^{13} · 3^4 · 5 · 7 · 17$	$2^{16} · 3^5 · 5^2 · 7 · 17$
5	$2^{21} · 3^5 · 5^2 · 7 · 17 · 31$	$2^{21} · 3^6 · 5^3 · 7 · 11 · 17$	$2^{25} · 3^6 · 5^2 · 7 · 11 · 17 · 31$

§ 3. Cremona lattices and Todorov lattices

A lattice is a free finitely generated  $\mathbb{Z}$ -module  $M$  together with a bilinear form  $b: M \times M \rightarrow \mathbb{Z}$ ; the lattice is even if  $b(x, x) \in 2\mathbb{Z}$  for all  $x \in M$ . A lattice  $M$  is nondegenerate if  $\text{Ad } b: M \rightarrow \text{Hom}(M, \mathbb{Z})$  is injective, and unimodular if  $\text{Ad } b$  is an isomorphism. If  $M$  is a nondegenerate even lattice, the discriminant-form of  $M$  is the finite quadratic form  $(G_M, q_M)$  where  $G_M = \text{Coker}(\text{Ad } b)$  and  $q_M$  is induced by the natural embedding of  $\text{Hom}(M, \mathbb{Z})$  into  $M \otimes \mathbb{Q}$ . We let  $O(G_M)$  denote the orthogonal group of  $(G_M, q_M)$ .

We single out several lattices for special attention. The  $K3$  lattice  $A$  is an even unimodular lattice of signature  $(3, 19)$ ; it is unique up to isomorphism by a theorem of Milnor [15; Section II.5]. The Cremona lattice  $M_n$  (for  $0 \leq n \leq 10$ ) is the lattice generated by the elements  $\lambda, e_1, \dots, e_n$ , where  $\lambda \cdot \lambda = 2, \lambda \cdot e_i = 0, e_i \cdot e_j = -2\delta_{ij}$ .

Let  $E_k$  be the lattice generated by the elements  $e_1, \dots, e_k$  where  $e_i \cdot e_j = -2\delta_{ij}$ . The double point lattice  $L_{\alpha,k}$  is the lattice defined as the saturation  $E_k \otimes \mathbb{Q} \cap j^{-1}(A)$  for some embedding  $j: E_k \hookrightarrow A$ , where the index  $[L_{\alpha,k}: E_k]$  is equal to  $2^\alpha$ . The double point lattice  $L_{\alpha,k}$  exists if and only if  $\alpha \leq 5$  and  $2^{4-\alpha}(2^\alpha - 1) \leq k \leq \alpha + 11$  and it is uniquely determined by  $(\alpha, k)$  up to isometries [20, Sections 1 and 2]. The Todorov lattice  $M_{\alpha,k}$ , defined for  $\alpha \leq 4, k \geq 9$  and  $2^{4-\alpha}(2^\alpha - 1) \leq k \leq \alpha + 11$  but  $(\alpha, k) \neq (1, 9)$ , is the lattice generated by  $L_{\alpha,k}$  and the elements  $\lambda$  and  $\mu$ , where  $\lambda \cdot \lambda = 2k - 16, \lambda \cdot L_{\alpha,k} = 0$  and  $\mu = (\lambda + \sum e_i)/2$ . For details, see [20, Section 6]. (The case  $(\alpha, k) = (5, 16)$  was also considered in [20], but we stress that in this paper, all Todorov lattices have  $k \leq 15$ .) We let  $(G_n, q_n)$  and  $(G_{\alpha,k}, q_{\alpha,k})$  denote the discriminant-forms of  $M_n$  and  $M_{\alpha,k}$  respectively, and let  $O(G_n)$  and  $O(G_{\alpha,k})$  denote the orthogonal groups of these discriminant-forms.

**Proposition (3.1).** *The normal form decompositions and the orders of*

the orthogonal groups of  $(G_n, q_n)$  and  $(G_{\alpha,k}, q_{\alpha,k})$  are those shown in Tables 3 and 4, respectively.

Table 3. Normal form decomposition of  $(G_n, q_n)$ , and the order of its orthogonal group

$n$	$(G_n, q_n)$	$ O(G_n, q_n) $
0	$w_{2,1}^1$	$ O^+(0, 2)  = 1$
1	$w_{2,1}^1 \oplus w_{2,1}^7$	$ O^+(0, 2)  = 1$
2	$u_1 \oplus w_{2,1}^7$	$ O^+(2, 2)  = 2$
3	$u_1 \oplus (w_{2,1}^7)^{\oplus 2}$	$2 \cdot  Sp(2, 2)  = 2^2 \cdot 3$
4	$u_1 \oplus u_1 \oplus w_{2,1}^1$	$ O^-(4, 2)  = 2^3 \cdot 3 \cdot 5$
5	$u_1 \oplus u_1 \oplus w_{2,1}^1 \oplus w_{2,1}^7$	$2^4 \cdot  O^-(4, 2)  = 2^7 \cdot 3 \cdot 5$
6	$u_1^{\oplus 2} \oplus u_1 \oplus w_{2,1}^7$	$ O^-(6, 2)  = 2^7 \cdot 3^4 \cdot 5$
7	$u_1^{\oplus 2} \oplus u_1 \oplus (w_{2,1}^7)^{\oplus 2}$	$2 \cdot  Sp(6, 2)  = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$
8	$u_1^{\oplus 4} \oplus w_{2,1}^1$	$ O^+(8, 2)  = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$
9	$u_1^{\oplus 4} \oplus w_{2,1}^1 \oplus w_{2,1}^7$	$2^8 \cdot  O^+(8, 2)  = 2^{21} \cdot 3^3 \cdot 5^2 \cdot 7$
10	$u_1^{\oplus 5} \oplus w_{2,1}^7$	$ O^+(10, 2)  = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$

Table 4. Normal form decomposition of  $(G_{\alpha,k}, q_{\alpha,k})$ , and the order of its orthogonal group

$\alpha$	$k$	$(G_{\alpha,k}, q_{\alpha,k})$	$ O(G_{\alpha,k}, q_{\alpha,k}) $
0	9	$u_1^{\oplus 4}$	$ O^+(8, 2)  = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$
0	10	$u_1^{\oplus 3} \oplus u_1 \oplus w_{2,2}^3$	$2 \cdot  Sp(8, 2)  = 2^{17} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$
0	11	$u_1^{\oplus 4} \oplus u_1 \oplus w_{3,1}^{-1}$	$ O^-(10, 2)  \cdot 2 = 2^{22} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 17$
1	10	$u_1^{\oplus 2} \oplus u_1 \oplus w_{2,2}^3$	$2 \cdot  Sp(6, 2)  = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$
1	11	$u_1^{\oplus 3} \oplus u_1 \oplus w_{3,1}^{-1}$	$ O^-(8, 2)  \cdot 2 = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
1	12	$u_1^{\oplus 3} \oplus u_1 \oplus w_{2,3}^5$	$2^9 \cdot  O^-(8, 2)  = 2^{22} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
2	12	$u_1^{\oplus 2} \oplus u_1 \oplus w_{2,3}^5$	$2^7 \cdot  O^-(6, 2)  = 2^{14} \cdot 3^4 \cdot 5$
2	13	$u_1^{\oplus 3} \oplus u_1 \oplus w_{5,1}^{-1}$	$ O^-(8, 2)  \cdot 2 = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
3	14	$u_1^{\oplus 3} \oplus w_{2,2}^5 \oplus w_{3,1}^1$	$2 \cdot  Sp(6, 2)  \cdot 2 = 2^{11} \cdot 3^4 \cdot 5 \cdot 7$
4	15	$u_1^{\oplus 3} \oplus w_{7,1}^1$	$ O^+(6, 2)  \cdot 2 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$

*Proof.* Since  $M_n$  is given in diagonal form, it is easy to see that  $(G_n, q_n)$  is isomorphic to  $w_{2,1}^1 \oplus (w_{2,1}^7)^{\oplus n}$ . The normal forms shown in Table 3 now follow directly from the relations in  $\text{qu}(\mathbf{Z})$  given in Proposition (1.2) (5) (cf. the proof of Proposition (1.3)).

[20; Proposition (6.2)] shows that  $(G_{\alpha,k}, q_{\alpha,k}) \cong z_l^\alpha \oplus (G'_{\alpha,k}, q'_{\alpha,k})$ ,

with  $(G'_{\alpha,k}, q'_{\alpha,k})$  a special 2-elementary form of rank  $2s$ , where

$$(2s, l, a) = \begin{cases} (k-2\alpha-1, k-8, 2) & \text{if } k \text{ is odd} \\ (k-2\alpha-2, 2k-16, k-7) & \text{if } k \text{ is even } (k < 16). \end{cases}$$

If  $k \neq 14$ ,  $l$  is a prime power so that  $z_l^\alpha$  is indecomposable; if  $k=14$  then  $z_l^\alpha = z_{12}^7 \cong w_{2,2}^5 \oplus w_{3,1}^1$ .

Since  $(G'_{\alpha,k}, q'_{\alpha,k})$  is special and 2-elementary of rank  $2s$ , we have  $(G'_{\alpha,k}, q'_{\alpha,k}) \cong u_1^{\oplus(s-m)} \oplus v_1^{\oplus m}$  with  $m=0$  or  $1$ : it remains to determine  $m$ . To accomplish this, we use Milgram's theorem [15; Appendix 4], which says that the first signature invariant  $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$  (which was introduced in the proof of Proposition (1.4)) coincides mod 8 with the signature of  $M_{\alpha,k}$ . The signature of  $M_{\alpha,k}$  is  $1-k$ ; on the other hand,  $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$  can be computed from Table 1. We have carried out this computation in Table 5; the conclusion we draw is that to get  $\sigma_1(G_{\alpha,k}, q_{\alpha,k}) \equiv 1-k \pmod 8$ , we must have  $m=0$  for  $k=9$  and  $k \geq 14$ , while  $m=1$  for  $10 \leq k \leq 13$ . This gives the normal forms shown in Table 4.

The orders of the orthogonal groups are computed as follows: by Example (2.1) (1) we only need to compute the orders for the  $p$ -Sylow subgroups, and these orders when  $p$  is odd follow from Example (2.1) (2). For the 2-Sylow subgroups, we use the normal form decomposition together with Corollary (2.4) and Table 2 to obtain the orders given in Tables 3 and 4. Q.E.D.

Table 5. Computation of  $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$

$k$	$(G_{\alpha,k}, q_{\alpha,k})$	$\sigma_1(G_{\alpha,k}, q_{\alpha,k})$
9	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m}$	$4m$
10	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{2,2}^3$	$4m+3$
11	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{3,1}^{-1}$	$4m+2$
12	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{2,3}^5$	$4m+1$
13	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{5,1}^{-1}$	$4m+0$
14	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{2,2}^5 \oplus w_{3,1}^1$	$4m+5-2$
15	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w_{7,1}^1$	$4m-6$

A map  $\phi: M \rightarrow L$  between nondegenerate lattices is called a *primitive embedding* if  $\phi$  preserves the bilinear forms, the kernel of  $\phi$  is trivial, and the cokernel of  $\phi$  is free. A surjective primitive embedding is called an *isometry*. The group of self-isometries of a nondegenerate lattice  $L$  is called the *orthogonal group of  $L$*  and denoted by  $O(L)$ . Any  $\gamma \in O(L)$  induces a natural automorphism of  $(G_L, q_L)$ , denoted by  $G_\gamma \in O(G_L)$ .

The *signature* of a nondegenerate lattice  $L$  is the pair  $(r_+, r_-)$  describing the number of positive and negative eigenvalues of the induced real quadratic form on  $L \otimes \mathbb{R}$ . A *(positive) sign structure* (cf. [13]) on a nondegenerate lattice  $L$  of signature  $(r_+, r_-)$  is a choice of one of the connected components of the set of oriented  $r_+$ -planes in  $L \otimes \mathbb{R}$  on which the form is positive-definite; the sign structure containing the oriented plane  $\nu$  is denoted by  $[\nu]$ .  $O_-(L)$  is the subgroup of  $O(L)$  of isometries preserving a sign structure; it can also be described as

$$O_-(L) = \{ \gamma \in O(L) \mid \det \gamma = \text{spin } \gamma \}$$

where  $\text{spin } \gamma$  denotes the (real) spinor norm.

**Theorem (3.2).** *Suppose that  $2 \leq n \leq 10$ .*

- (1) *Let  $\phi_1, \phi_2: M_n \rightarrow \Lambda$  be two primitive embeddings of the Cremona lattice into the K3 lattice. Then there is some  $\gamma \in O_-(\Lambda)$  such that  $\gamma \circ \phi_1 = \phi_2$ .*
- (2) *Let  $\phi: M_n \rightarrow \Lambda$  be a primitive embedding, and let  $N_n$  be the orthogonal complement of the image of  $\phi$ . If  $\psi_1, \psi_2: N_n \rightarrow \Lambda$  are two primitive embeddings, then there is some  $\gamma \in O_-(\Lambda)$  such that  $\gamma \circ \psi_1 = \psi_2$ .*

*Proof.* (1) Since  $N_n$  has signature  $(2, 19 - n)$ , it is indefinite and has rank at least 3. Since  $n \geq 2$ , by Proposition (3.1)  $(G_n, q_n)$  contains an orthogonal summand of type  $u_1$  or  $v_1$ , and the  $p$ -Sylow subgroup of  $G_n$  is cyclic for  $p \neq 2$ . The statement now follows from Theorem (A.1) of [20] (which is a variant of a theorem of Nikulin [21]).

(2) We again use Theorem (A.1) of [20]. This time we must check that  $M_n$  (which has signature  $(1, n)$ ) is indefinite and of rank at least 3; this holds for  $n \geq 2$ . By a standard argument [22; Section 1.6], there is an isomorphism  $(G_{N_n}, q_{N_n}) \cong (G_{M_n}, -q_{M_n}) = (G_n, -q_n)$ . Since the isomorphism types of  $u_1$  and  $v_1$  do not change when the signs on the quadratic forms are reversed, Proposition (3.1) again implies that  $(G_{N_n}, q_{N_n})$  contains an orthogonal summand of type  $u_1$  or  $v_1$ , and that the  $p$ -Sylow subgroup of  $G_{N_n}$  is cyclic for  $p \neq 2$ ; the theorem follows. Q.E.D.

For Todorov lattices, the statement analogous to Theorem (3.2) (1) was proved in [20; Theorem 6.3].

**Theorem (3.3).** *Let  $M_{\alpha,k}$  be a Todorov lattice, let  $\phi: M_{\alpha,k} \rightarrow A$  be an embedding into the K3 lattice, and let  $N_{\alpha,k}$  be the orthogonal complement of the image of  $\phi$ . If  $\psi_1, \psi_2: N_{\alpha,k} \rightarrow A$  are two primitive embeddings, then there is some  $\gamma \in O_-(A)$  such that  $\gamma \circ \psi_1 = \psi_2$ .*

*Proof.*  $M_{\alpha,k}$  has signature  $(1, k-1)$ ; since  $k \geq 9$ ,  $M_{\alpha,k}$  is therefore indefinite and has rank at least 3. Since  $(G_{N_{\alpha,k}}, q_{N_{\alpha,k}}) \cong (G_{\alpha,k}, -q_{\alpha,k})$ , Proposition (3.1) implies that  $(G_{N_{\alpha,k}}, q_{N_{\alpha,k}})$  contains an orthogonal summand of type  $u_1$  or  $v_1$ , and that the  $p$ -Sylow subgroup of  $G_{N_{\alpha,k}}$  is cyclic for  $p \neq 2$ . We may then apply Theorem (A.1) of [20] one more time, and the theorem follows. Q.E.D.

**§ 4. The moduli of double sextics with  $n$  nodes**

A double sextic with  $n$  nodes is a double cover  $X$  of  $\mathbb{P}^2$  branched along a sextic curve  $C$  in  $\mathbb{P}^2$  which has exactly  $n$  nodes as its singularities. Any double sextic  $X$  with  $n$  nodes is a K3 surface with  $n$  ordinary double points (lying over the nodes of  $C$ ). Let  $\mu: S \rightarrow X$  be the minimal desingularization of  $X$ , and let  $\lambda: V \rightarrow \mathbb{P}^2$  be the blow up of the  $n$  nodes of  $C$ . We then get a Cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{\lambda} & \mathbb{P}^2 \end{array}$$

Let  $M_n(X)$  be the sublattice of  $H^2(S, \mathbb{Z})$  generated by  $c_1(\mathcal{L}), c_1(E_1), \dots, c_1(E_n)$ , where  $\mathcal{L} = (g \circ \lambda)^* \mathcal{O}_{\mathbb{P}^2}(1)$  and  $\{E_i\}$  are the exceptional curves of  $\mu$ . Then  $M_n(X) \otimes \mathbb{Q}$  coincides with  $g^* H^2(V, \mathbb{Q})$ . The intersection numbers are given by  $\mathcal{L} \cdot \mathcal{L} = 2, \mathcal{L} \cdot E_i = 0, E_i \cdot E_j = -2\delta_{ij}$ , so that  $M_n(X)$  is isomorphic to the Cremona lattice  $M_n$ .

**Lemma (4.1).** *Let  $X$  be a double sextic with  $n$  nodes, and let  $S$  be its minimal desingularization. Then the inclusion  $M_n(X) \subset H^2(S, \mathbb{Z})$  is a primitive embedding if and only if the branch curve  $C$  is irreducible.*

*Proof.* Let  $\iota: S \rightarrow S$  be the involution such that  $S/\iota = V$ . We first claim that if  $x \in (M_n(X) \otimes \mathbb{Q}) \cap H^2(S, \mathbb{Z})$ , then  $\iota^*(x) = x$  and  $2x \in M_n(X)$ . For suppose that  $kx \in M_n(X)$  for some  $k \in \mathbb{Z}$ . Then  $k\iota^*(x) = \iota^*(kx) = kx$  so that  $k(\iota^*(x) - x) = 0$ ; since  $H^2(S, \mathbb{Z})$  is torsion free, we have  $\iota^*(x) = x$ . Moreover,  $g^* g_*(x) = x + \iota^*(x) = 2x \in M_n(X)$ , proving our claim.

Let  $C = \sum_i C_i$  be the decomposition into irreducible components, and let  $\tilde{C}_i$  be the proper transform of  $C_i$  on  $V$ . Then it is easy to see that

$\mathcal{O}_S(g^*(\tilde{C}_I) + \sum_{i \in I} E_i + \sum_{j \in J} 2E_j) = \mathcal{L}^{\otimes d}$ , where  $d = \deg C_I$ ,  $I$  is the set of indices corresponding to the intersection points  $C_I \cap (\sum_{k \neq I} C_k)$ , and  $J$  is the set of indices corresponding to the double points of  $C_I$ . If  $C$  is reducible, then  $C_I \cap (\sum_{k \neq I} C_k) \neq \emptyset$ ; since  $g^*(\tilde{C}_I) = 2\tilde{C}'_I$  for some curve  $\tilde{C}'_I$  on  $S$ , we have  $c_1(\tilde{C}'_I) \notin M_n(X)$  but  $c_1(2\tilde{C}'_I) \in M_n(X)$ , so that the inclusion  $M_n(X) \subset H^2(S, \mathbf{Z})$  is not a primitive embedding (its cokernel is not free).

Suppose now that  $C$  is irreducible, and take any  $x \in H^2(S, \mathbf{Z})$  such that  $2x \in M_n(X)$ . It suffices to show that  $x \in M_n(X)$ . By replacing (if necessary)  $x$  by  $x + c_1(g^*(\mathcal{H}))$  for a sufficiently ample line bundle  $\mathcal{H}$ , we may assume that  $x = c_1(D')$  for some effective divisor  $D'$ . Since  $\iota^*(x) = x$ , the involution  $\iota^*$  acts on the complete linear system  $|D'|$ . Now  $\iota^*$  has finite order, so its action on  $|D'|$  must have fixed points. In other words, there is an effective divisor  $D \in |D'|$  such that  $\iota(D) = D$ .

Let  $C'$  be the branch divisor of  $g$  in  $S$ . Since  $\iota(D) = D$ ,  $D$  can be written as a sum of irreducible divisors as follows:

$$D = aC' + \sum_i b_i(F_i + \iota(F_i)) + \sum_k c_k G_k.$$

Here  $F_i$  and  $G_k$  denote irreducible divisors such that  $\iota(F_i) \neq F_i$  and  $\iota(G_k) = G_k$  for all  $i$  and  $k$ . Clearly,  $c_1(F_i + \iota(F_i)) \in M_n(X)$  and  $c_1(G_k) \in M_n(X)$ . It is easy to check that  $\mathcal{L}^{\otimes 6} = \mathcal{O}_S(2C' + \sum_{j=1}^n 2E_j)$  which implies that  $c_1(C') \in M_n(X)$ . Thus,  $x = c_1(D) \in M_n(X)$ . Q.E.D.

A consequence of Lemma (4.1) is the existence of a primitive embedding of the Cremona lattice  $M_n$  into the K3 lattice  $A$  for  $0 \leq n \leq 10$ , since there exist irreducible sextic curves with exactly  $n$  nodes in those cases. We fix once and for all a primitive embedding of  $M_n$  into  $A$ , and identify  $M_n$  with its image in  $A$ . We let  $N_n$  be the orthogonal complement of  $M_n$  in  $A$ , and fix a positive sign structure  $[v_n]$  on  $N_n$ . The *period space*  $D_n$  is then defined by

$$D_n = \{ \omega \in P(N_n \otimes \mathbf{C}) \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0, \text{Re}(\omega) \wedge \text{Im}(\omega) \in [v_n] \}.$$

(The last condition ensures that  $D_n$  is connected.) The *integral automorphism group* of  $D_n$  is the group  $\text{Aut}_{\mathbf{Z}}(D_n) = O_-(N_n)/(\pm 1)$ .

A *K3 surface of Cremona type  $n$*  is a pair  $(X, \mathcal{L})$  consisting of a K3 surface  $X$  with  $n$  ordinary double points and a line bundle  $\mathcal{L}$  on  $X$  with  $\mathcal{L} \cdot \mathcal{L} = 2$  whose associated linear system is free, and maps  $X$  to  $\mathbf{P}^2$  with branch locus an irreducible sextic curve with  $n$  nodes. If  $(X, \mathcal{L})$  is a K3 surface of Cremona type  $n$ , and  $\mu: S \rightarrow X$  is the minimal desingularization, a *special marking* of  $(X, \mathcal{L})$  is an isometry  $\phi: H^2(S, \mathbf{Z}) \rightarrow A$  together with an isomorphism  $\psi: \text{Sing } X \rightarrow \{1, \dots, n\}$  with the following properties:

- (1)  $c_1(\mu^* \mathcal{L}) = \phi^{-1}(\lambda)$ , and  $c_1(\mu^{-1}(P)) = \phi^{-1}(e_{\psi(P)})$  for each  $P \in \text{Sing } X$ ;

- (2) for any nonzero holomorphic 2-form  $\omega$  on  $S$ ,  $\text{Re}(\phi(\omega)) \wedge \text{Im}(\phi(\omega)) \in [\nu_n]$  (regarding  $\phi(\omega)$  as an element of  $N_n \otimes \mathbb{C}$ );
- (3) If  $x$  in  $\text{NS}(S)$  satisfies the conditions that  $x \cdot x = -2$  and  $x \cdot \phi^{-1}(\lambda) = 0$ , then  $x$  is in  $\phi^{-1}(M_n)$ . (Here,  $\text{NS}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$  is the Néron-Severi group.)
- (4) There are no elements  $y$  in  $\text{NS}(S)$  with  $y \cdot y = 0$  and  $y \cdot \phi^{-1}(\lambda) = 1$ .

**Lemma (4.2).** *Every K3 surface of Cremona type  $n$  has a special marking.*

*Proof.* Choose an isomorphism  $\psi: \text{Sing } X \rightarrow \{1, \dots, n\}$  and an isometry  $\eta: M_n(X) \rightarrow M_n$  such that  $\eta^{-1}(\lambda) = c_1(\mu^* \mathcal{L})$  and  $\eta^{-1}(e_{\psi(P)}) = c_1(\mu^{-1}(P))$ . Also choose an isometry  $\phi': H^2(S, \mathbb{Z}) \rightarrow A$  such that  $\phi'[c_1(\mu^* \mathcal{L}) \wedge \text{Re}(\omega) \wedge \text{Im}(\omega)] = [\lambda \wedge \nu_n]$  for a nonzero holomorphic 2-form  $\omega$  on  $S$ . Since the branch locus of the map  $f: X \rightarrow \mathbb{P}^2$  is irreducible, Lemma (4.1) guarantees that  $\phi' \circ \eta^{-1}$  determines a primitive embedding of  $M_n$  into  $A$ ; by Theorem (3.2) (1), there is some  $\gamma \in O_-(A)$  with  $\gamma \circ \phi' \circ \eta^{-1} = 1_{M_n}$ . Let  $\phi = \gamma \circ \phi'$ ; then  $\phi^{-1}(\lambda) = c_1(\mu^* \mathcal{L})$  and  $\phi^{-1}(e_{\psi(P)}) = c_1(\mu^{-1}(P))$  for each  $P \in \text{Sing } X$ . Since  $\gamma$  preserves the sign structure  $[\lambda \wedge \nu_n]$  on  $A$ ,  $[\lambda \wedge \text{Re}(\phi(\omega)) \wedge \text{Im}(\phi(\omega))] = \phi[c_1(\mu^* \mathcal{L}) \wedge \text{Re}(\omega) \wedge \text{Im}(\omega)] = [\lambda \wedge \nu_n]$  so that  $\text{Re}(\phi(\omega)) \wedge \text{Im}(\phi(\omega)) \in [\nu_n]$  as sign structures on  $N_n$ .

If  $x$  in  $\text{NS}(S)$  satisfies the conditions  $x \cdot x = -2$  and  $x \cdot \phi^{-1}(\lambda) = 0$ , then  $\pm x$  is the first Chern class of an effective divisor  $E$  with  $\mathcal{L} \cdot E = 0$ . The support of  $E$  is contained in  $\mu^{-1}(\text{Sing } X)$ , so that  $\pm x = c_1(E)$  is in the inverse image  $\phi^{-1}(M_n)$ .

Suppose  $y$  in  $\text{NS}(S)$  satisfies the conditions  $y \cdot y = 0$  and  $y \cdot \phi^{-1}(\lambda) = 1$ . By Riemann-Roch,  $\pm y$  is the first Chern class of an effective divisor  $F$  which moves in a linear system of (projective) dimension at least 1; since  $y \cdot \mu^* \mathcal{L} > 0$  and  $\mu^* \mathcal{L}$  is nef, we have  $y = c_1(F)$ . Since irreducible curves of negative self-intersection on a K3 surface do not move, there must be some irreducible component  $F_0$  of some member of the complete linear system  $|F|$  with  $F_0 \cdot F_0 \geq 0$ . Now  $F_0 \cdot \mu^* \mathcal{L} \neq 0$ , for otherwise  $F_0$  would have self-intersection  $-2$ ; hence,  $F_0 \cdot \mu^* \mathcal{L} = 1$ . The Hodge index theorem now implies that  $F_0 \cdot F_0 \leq 0$ , so that  $F_0$  is an effective irreducible curve with  $F_0 \cdot F_0 = 0$  and  $F_0 \cdot \mu^* \mathcal{L} = 1$ . By Mayer [14], the linear system  $|\mu^* \mathcal{L}|$  cannot be free (it has the form  $\mu^* \mathcal{L} = \mathcal{O}_S(2F_0 + C)$  for some rational curve  $C$  which serves as its base locus), a contradiction. Hence, no such  $y$  exists. Q.E.D.

Note that if  $\gamma \in O(A)$  satisfies  $\gamma(\lambda) = \lambda$  and  $\gamma(M_n) = M_n$  then  $\gamma|_{N_n}$  preserves the sign structure  $[\nu_n]$  if and only if  $\gamma \in O_-(A)$ . Thus, if  $(\phi, \psi)$  is a special marking of  $(X, \mathcal{L})$ , it follows directly from the definitions that  $(\phi', \psi')$  is a special marking of the same  $(X, \mathcal{L})$  if and only if  $(\phi', \psi') = (\gamma \circ \phi, \sigma \circ \psi)$  for some  $(\gamma, \sigma) \in \tilde{\Gamma}_n$ , where

$$\tilde{\Gamma}_n = \{(\gamma, \sigma) \in O_-(A) \times \mathfrak{S}_n \mid \gamma(\lambda) = \lambda, \gamma(e_i) = e_{\sigma(i)}\}$$

and  $\mathfrak{S}_n$  denotes the symmetric group on  $n$  letters.

If  $(\gamma, \sigma) \in \tilde{\Gamma}_n$  then  $\gamma|_{N_n}$  preserves the sign structure  $[\nu_n]$  so that  $\gamma|_{N_n}$  acts on  $D_n$ ; we define  $\Gamma_n = \text{Image}(\tilde{\Gamma}_n \rightarrow \text{Aut}_{\mathbf{Z}}(D_n))$ . The set of special markings of  $(X, \mathcal{L})$  thus determines a point in  $D_n/\Gamma_n$ .

Define the open set  $D_n^\circ$  of  $D_n$  as follows:

$$D_n^\circ = \{\omega \in D_n \mid \text{for all } \xi, \eta \text{ in } A, \text{ if } \xi \cdot \xi = -2, \xi \cdot \lambda = 0, \text{ and } \xi \notin M_n \text{ then } \xi \cdot \omega \neq 0; \text{ if } \eta \cdot \eta = 0, \text{ and } \eta \cdot \lambda = 1, \text{ then } \eta \cdot \omega \neq 0\}.$$

Note that  $D_n^\circ$  is stable under the action of  $\Gamma_n$ , and that a special marking of  $(X, \mathcal{L})$  determines a point in  $D_n^\circ$  (the marking sends  $x, y$  to  $\xi, \eta$ ).

**Theorem (4.3).**  *$D_n^\circ/\Gamma_n$  is a coarse moduli space for K3 surfaces of Cremona type  $n$ , and for irreducible sextic curves in  $\mathbf{P}^2$  with  $n$  nodes.*

*Proof.* First note that there is a one-to-one correspondence between K3 surfaces of Cremona type  $n$  and irreducible sextic curves in  $\mathbf{P}^2$  with  $n$  nodes obtained by sending the surface  $(X, \mathcal{L})$  to the branch locus of the map defined by the linear system  $|\mathcal{L}|$  (and inversely sending a sextic curve to the double cover of  $\mathbf{P}^2$  branched on that curve.)

Suppose that  $(X_i, \mathcal{L}_i)$  for  $i=1, 2$  are K3 surfaces of Cremona type  $n$  which are assigned to the same point in  $D_n^\circ/\Gamma_n$ , and let  $\mu_i: S_i \rightarrow X_i$  be the minimal desingularizations. There are then special markings  $\phi_i: H^2(S_i, \mathbf{Z}) \rightarrow A$ ,  $\psi_i: \text{Sing } X_i \rightarrow \{1, \dots, n\}$  and we may assume (by changing one of the markings by an element of  $\tilde{\Gamma}_n$  if necessary) that  $\phi = \phi_2^{-1} \circ \phi_1: H^2(S_1, \mathbf{Z}) \rightarrow H^2(S_2, \mathbf{Z})$  is an isomorphism of Hodge structures; note that  $\phi(c_1(\mu_1^*(\mathcal{L}_1))) = c_1(\mu_2^*(\mathcal{L}_2))$ . Let  $\psi = \psi_2^{-1} \circ \psi_1: \text{Sing } X_1 \rightarrow \text{Sing } X_2$  be the induced isomorphism. An easy computation shows that  $\phi(c_1(\mu_1^{-1}(P))) = c_1(\mu_2^{-1}(\psi(P)))$  so that  $\phi$  sends the effective curves  $\mu_1^{-1}(P)$  to effective curves  $\mu_2^{-1}(\psi(P))$ .

We now use the weakly polarized global Torelli theorem [19; p. 329] (cf. also [24]): since  $\phi(c_1(\mu_1^*(\mathcal{L}_1))) = c_1(\mu_2^*(\mathcal{L}_2))$  and since each irreducible curve  $\mu_1^{-1}(P)$  which is contracted by  $\mu_1$  is mapped to an effective curve by  $\phi$ , there is an isomorphism  $\Phi: X_2 \rightarrow X_1$  such that  $\Phi^* = \phi$ . Since  $\Phi^*(\mathcal{L}_1) = \mathcal{L}_2$ ,  $\Phi$  induces an isomorphism between the pairs  $(X_i, \mathcal{L}_i)$  for  $i=1, 2$ ; we conclude that the natural map from the moduli space of such pairs to  $D_n^\circ/\Gamma_n$  is injective.

To prove that the map is surjective, take an arbitrary point  $\omega \in D_n^\circ$ . We use the surjectivity of the period map for algebraic K3 surfaces in the form given in [19; p. 325] (and due essentially to Kulikov [11]): there is a K3 surface with rational double points  $X$ , an ample line bundle  $\mathcal{L}$  on  $X$  and

an isometry  $\phi: H^2(S, \mathbb{Z}) \rightarrow A$  with  $\phi^{-1}(\omega) \in H^{2,0}(S)$  and  $\phi^{-1}(\lambda) = c_1(\mu^*(\mathcal{L}))$  where  $\mu: S \rightarrow X$  is the minimal desingularization. By composing  $\phi$  with reflections in some of the classes  $e_i$  if necessary, we may assume that each  $\phi^{-1}(e_i)$  is the first Chern class of an effective divisor.

If  $E$  is the class of an irreducible curve contracted by  $\mu$ , then  $c_1(E)$  lies in  $\phi^{-1}(M_n)$  (since we selected a point from  $D_n^\circ$ ) so that  $c_1(E)$  must coincide with  $\phi^{-1}(e_i)$  for some  $i \in \{1, \dots, n\}$ . Hence,  $X$  has exactly  $n$  nodes; let  $\psi: \text{Sing } X \rightarrow \{1, \dots, n\}$  be the isomorphism compatible with  $\phi$ .

Suppose that the linear system  $|\mathcal{L}|$  is not free; then  $|\mu^*\mathcal{L}|$  is also not free. Since  $\mu^*\mathcal{L}$  is nef, by Mayer [14],  $\mu^*\mathcal{L}$  has the form  $\mu^*\mathcal{L} = \mathcal{O}_S(2F + C)$  for some elliptic curve  $F$  and rational curve  $C$  with  $F \cdot \mu^*\mathcal{L} = 1$ ; this cannot happen since we selected a point from  $D_n^\circ$ .

Hence,  $|\mathcal{L}|$  is free,  $(X, \mathcal{L})$  is a  $K3$  surface of Cremona type  $n$ , and  $(\phi, \psi)$  is a special marking sending it to the class of the chosen point of  $D_n^\circ$ .  
 Q.E.D.

§ 5. Degrees of period maps

In the moduli problems for sextic curves with nodes studied in the last section, and in the moduli problem for Todorov surfaces studied in [20], we have encountered families of polarized  $K3$  surfaces whose moduli spaces have a similar type of description. In each case, there is a lattice  $M$  primitively embedded in the  $K3$  lattice  $A$  with orthogonal complement  $N$  such that  $M \otimes \mathbb{Q}$  is generated by classes  $\lambda$  and  $\{e_i | i \in I\}$  with  $\lambda \cdot \lambda > 0$ ,  $\lambda \cdot e_i = 0$ , and  $e_i \cdot e_j = -2\delta_{ij}$ . The natural period space for such a moduli problem is

$$D = \{\omega \in P(N \otimes \mathbb{C}) | \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0, \text{Re}(\omega) \wedge \text{Im}(\omega) \in [\nu]\}$$

once a sign structure  $[\nu]$  has been fixed on  $N$ ;  $D$  classifies polarized Hodge structures on  $N$  with the given sign structure. The group  $\text{Aut}_{\mathbb{Z}}(D) = O_-(N)/(\pm 1)$  acts on  $D$  and the quotient  $D/\text{Aut}_{\mathbb{Z}}(D)$  parametrizes equivalence classes of Hodge structures on  $N$ , where two Hodge structures are equivalent if there is an automorphism of  $N$  preserving the bilinear form and sign structure which induces an isomorphism from one Hodge structure to the other.

Associated to  $M$  is a group  $\mathfrak{S}$  which is the set of permutations  $\sigma$  of  $I$  such that the automorphism  $\lambda \mapsto \lambda, e_i \mapsto e_{\sigma(i)}$  of  $M \otimes \mathbb{Q}$  induces an automorphism of  $M$ ; this formula defines a natural homomorphism  $j: \mathfrak{S} \rightarrow O_-(M)$ . If we define

$$\tilde{\Gamma} = \{(\gamma, \sigma) \in O_-(A) \times \mathfrak{S} | \gamma(\lambda) = \lambda, \gamma(e_i) = e_{\sigma(i)}\}$$

and  $\Gamma = \text{Image}(\tilde{\Gamma} \rightarrow \text{Aut}_{\mathbb{Z}}(D))$  then the moduli space of the  $K3$  surfaces

considered was described as  $M = D^\circ/\Gamma$ , where  $D^\circ$  is a certain open subset of  $D$ .

Note that in the case of K3 surfaces of Cremona type  $n$ ,  $M_n$  itself is generated by  $\lambda$  and  $\{e_i\}$ . Thus, the group  $\mathfrak{S}$  is the full symmetric group  $\mathfrak{S}_n$ ; we denote the corresponding group for K3 surfaces of Todorov type  $(\alpha, k)$  by  $\mathfrak{S}_{\alpha,k}$ . Note also that the index set  $I$  is  $\{1, \dots, n\}$  in the Cremona case, and  $\text{Sing } \Sigma_{\alpha,k}$  in the Todorov case, where  $\Sigma_{\alpha,k}$  is the fixed K3 surface of Todorov type  $(\alpha, k)$  chosen in [20; Section 2].

Since  $D/\text{Aut}_Z(D)$  is the natural classifying space for the Hodge structures being considered, there is a natural "period map"  $p: M \rightarrow D^\circ/\text{Aut}_Z(D)$ , which we will now study. In the two cases of primary interest to us, we denote the period maps by

$$p_n: M_n \rightarrow D_n^\circ/\text{Aut}_Z(D_n) \quad \text{and} \quad p_{\alpha,k}: M_{\alpha,k} \rightarrow D_{\alpha,k}^\circ/\text{Aut}_Z(D_{\alpha,k}).$$

**Theorem (5.1).** *Let  $p: M \rightarrow D^\circ/\text{Aut}_Z(D)$  be a period map built from  $M \subset A$  as above, and let  $\mathcal{J}$  be the image of the natural homomorphism  $i: \mathfrak{S} \rightarrow O(G_M)/(\pm 1)$ . Suppose that  $M$  satisfies:*

- (1) *for any two primitive embeddings  $\phi_1, \phi_2: M \rightarrow A$ , there is some  $\gamma \in O_-(A)$  such that  $\gamma \circ \phi_1 = \phi_2$ , and*
- (2) *the natural map  $O(M) \rightarrow O(G_M)$  is surjective.*

*Then  $\text{deg } p = |O(G_M)/(\pm 1)|/|\mathcal{J}|$ .*

*Proof.* Let  $N = M^\perp$ . We first claim that

$$\Gamma = \{\beta \in O_-(N) \mid \pm G_\beta \in \mathcal{J}\}/(\pm 1).$$

For if  $\beta \in O_-(N)$  then  $\pm\beta \in \Gamma$  if and only if there exists some  $(\gamma, \sigma) \in \tilde{\Gamma}$  with  $\gamma|_N = (-1)^k\beta$  for some  $k$ ; in that case,  $G_{(\gamma|M)} = i(\sigma)$  in  $O(G_M)$ . If we use the standard isomorphism  $(G_M, q_M) \cong (G_N, -q_N)$  to identify  $O(G_M)$  with  $O(G_N)$  then for a given  $\sigma \in \mathfrak{S}$  and  $\beta \in O_-(N)$ , the sum  $j(\sigma) \oplus (-1)^k\beta \in O_-(M \oplus N)$  comes from some  $\gamma \in O_-(A)$  if and only if  $G_{j(\sigma)} = (-1)^k G_\beta$ . Since  $i(\sigma) = G_{j(\sigma)}$ , a given  $\pm\beta$  thus lies in  $\Gamma$  if and only if  $\pm G_\beta \in \text{Image}(i)$ , proving the claim.

Since  $\text{deg } p = [\text{Aut}_Z(D): \Gamma]$ , it only remains to show that the natural map  $O_-(N) \rightarrow O(G_N)/(\pm 1)$  is surjective. Let  $\tau \in O(G_N) \cong O(G_M)$ ; there is then some  $\beta \in O(M)$  with  $G_\beta = \tau$ . Now  $-1_M \notin O_-(M)$  so that  $(-1)^k\beta \in O_-(M)$  for some  $k$ . Let  $\phi: M \subset A$  be the inclusion; then  $\phi \circ (-1)^k\beta: M \rightarrow A$  gives another primitive embedding. There is then some  $\gamma \in O_-(A)$  such that  $\gamma \circ \phi = \phi \circ (-1)^k\beta$ , in other words,  $\gamma|_M = (-1)^k\beta$ . But then  $G_{(\gamma|N)} = (-1)^k\beta$  in  $O(G_N)$ ; hence,  $(\gamma|_N) \in O_-(N)$  is the required element mapping to  $\pm\beta$ .

Q.E.D.

**Corollary (5.2).** *If  $2 \leq n \leq 10$ , the degree of the period map  $p_n: M_n \rightarrow D_n^o / \text{Aut}_{\mathbb{Z}}(D_n)$  is given by  $\deg p_n = |O(G_n)|/n!$ .*

These degrees are computed explicitly in Table 6.

Table 6. Degree of  $p_n$

$n$	$ O(G_n) $	$n!$	$\deg p_n$
2	2	2	1
3	$2^2 \cdot 3$	$2 \cdot 3$	2
4	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3$	5
5	$2^7 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 5$	$2^4$
6	$2^7 \cdot 3^4 \cdot 5$	$2^4 \cdot 3^2 \cdot 5$	$2^3 \cdot 3^2$
7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^2$
8	$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^3 \cdot 5$
9	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3 \cdot 5$
10	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	$2^{13} \cdot 3 \cdot 17 \cdot 31$

*Proof.* By Theorem (3.2) (1), the embedding  $M_n \subset A$  is unique up to action by  $O_-(A)$ . Moreover, [22; Theorem 1.14.2] together with our computation of  $(G_n, q_n)$  in Section 3 shows that the map  $O(M_n) \rightarrow O(G_n)$  is surjective, so we may apply Theorem (5.1). Since  $G_n$  is a 2-elementary group,  $-1$  is the trivial element of  $O(G_n)$  and  $|O(G_n)/(\pm 1)| = |O(G_n)|$ . Now  $G_n$  is generated by  $\lambda/2, e_1/2, \dots, e_n/2 \pmod{M_n}$ , so that every nontrivial permutation  $\sigma$  of  $\{e_i\}$  induces a nontrivial permutation of  $\{e_i/2 \pmod{M_n}\}$ . This implies that the map  $\mathfrak{S}_n \rightarrow O(G_n)$  is injective, so that  $\deg p_n = |O(G_n)|/|\mathfrak{S}_n|$ .

Q.E.D.

To compute the degree of  $p_{\alpha,k}$  we must study the finite group  $\mathfrak{S}_{\alpha,k}$ . Since  $k \leq 15$ , by [20; Proposition 6.1], the Todorov lattice  $M_{\alpha,k}$  is generated by the double point lattice  $L_{\alpha,k}$  and  $\mu = (\lambda + \sum e_p)/2$ . Thus, a permutation  $\sigma$  of  $\text{Sing } \Sigma_{\alpha,k}^4$  induces an automorphism of  $M_{\alpha,k}$  by the rule  $\lambda \mapsto \lambda, e_p \mapsto e_{\sigma(p)}$  if and only if  $\sigma$  induces an automorphism of the double point code  $\mathcal{C}_{\alpha,k}$ . Hence,  $\mathfrak{S}_{\alpha,k} \cong \text{Aut}(\mathcal{C}_{\alpha,k})$ ; by [20; Lemma 1.3] there is an exact sequence

<sup>4</sup> In [20],  $\Sigma_{\alpha,k}$  was introduced as a K3 surface with  $k$  ordinary double points  $p_1, \dots, p_k$  whose double point lattice is isomorphic to  $L_{\alpha,k}$  and which has a line bundle  $\mathcal{L}$  such that  $(\pi^*(\mathcal{L}))^2 = 2k - 16$  and  $(\pi^*(\mathcal{L}) + \sum \pi^{-1}(p_i))/2 \in H^2(S, \mathbb{Z})$ , where  $\pi: S \rightarrow \Sigma_{\alpha,k}$  is the minimal desingularization. An example of  $\Sigma_{\alpha,k}$  is given by a partial desingularization of the Kummer surface of a principally polarized abelian surface.

$$1 \longrightarrow (\mathfrak{S}_{2l-\alpha})^{2^{\alpha-1}} \times \mathfrak{S}_l \longrightarrow \mathfrak{S}_{\alpha,k} \longrightarrow \mathrm{GL}(\alpha, F_2) \longrightarrow 1$$

where  $l = k - 2^{4-\alpha}(2^\alpha - 1)$ . We have used this exact sequence to compute the orders of the groups  $\mathfrak{S}_{\alpha,k}$  in Table 7. (The computation involves the order of  $\mathrm{GL}(\alpha, F_2)$ , which can be found for example in [25; p. 81].)

Table 7. Order of  $\mathfrak{S}_{\alpha,k}$

$\alpha$	$k$	$l$	$ \mathfrak{S}_{\alpha,k} $
0	9	9	$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$
0	10	10	$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
0	11	11	$11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
1	10	2	$8! \cdot 2! = 2^8 \cdot 3^2 \cdot 5 \cdot 7$
1	11	3	$8! \cdot 3! = 2^8 \cdot 3^3 \cdot 5 \cdot 7$
1	12	4	$8! \cdot 4! = 2^{10} \cdot 3^3 \cdot 5 \cdot 7$
2	12	0	$6 \cdot (4!)^3 \cdot 0! = 2^{10} \cdot 3^4$
2	13	1	$6 \cdot (4!)^3 \cdot 1! = 2^{10} \cdot 3^4$
3	14	0	$168 \cdot (2!)^7 \cdot 0! = 2^{10} \cdot 3 \cdot 7$
4	15	0	$20160 \cdot (1!)^{15} \cdot 0! = 2^8 \cdot 3^2 \cdot 5 \cdot 7$

**Corollary (5.3).** *The degree of the period map  $p_{\alpha,k}: M_{\alpha,k} \rightarrow D_{\alpha,k}/\mathrm{Aut}_{\mathbb{Z}}(D_{\alpha,k})$  is given by*

$$\deg p_{\alpha,k} = \begin{cases} \frac{|O(G_{\alpha,k})|}{|\mathfrak{S}_{\alpha,k}|} & \text{if } (\alpha, k) = (0, 9) \text{ or } (1, 10) \\ \frac{|O(G_{\alpha,k})|}{2|\mathfrak{S}_{\alpha,k}|} & \text{otherwise.} \end{cases}$$

These degrees are computed explicitly in Table 8.

*Proof.* By [20; Theorem (6.3)] the embedding  $M_{\alpha,k} \subset A$  is unique up to action by  $O_-(A)$ . Moreover, [22; Theorem 1.14.2] together with our computation of  $(G_{\alpha,k}, q_{\alpha,k})$  in Section 3 shows that the natural map  $O(M_{\alpha,k}) \rightarrow O(G_{\alpha,k})$  is surjective, so that we may apply Theorem (5.1).

Let  $\mathcal{K}_{\alpha,k}$  be the kernel of the homomorphism  $i: \mathfrak{S}_{\alpha,k} \rightarrow O(G_{\alpha,k})/(\pm 1)$ . If  $\sigma \in \mathcal{K}_{\alpha,k}$  then  $\sigma$  acts as  $(-1)^m$  on  $G_{\alpha,k}$  for some  $m$ ; thus, there is some  $(\sigma, \gamma) \in \tilde{\Gamma}_{\alpha,k}$  with  $\gamma|_{N_{\alpha,k}} = (-1)^m$ . By [20; Lemma 7.5], the natural map  $\tilde{\Gamma}_{\alpha,k} \rightarrow O_-(N_{\alpha,k})$  is injective, and  $-1$  is in the image of that map if and only

Table 8. Degree of  $p_{\alpha,k}$

$\alpha$	$k$	$ O(G_{\alpha,k}) $	$ \mathfrak{S}_{\alpha,k} $	$\deg p_{\alpha,k}$
0	9	$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	$2^6 \cdot 3 \cdot 5$
0	10	$2^{17} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	$2^8 \cdot 3 \cdot 17$
0	11	$2^{22} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$2^{13} \cdot 3^2 \cdot 17$
1	10	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	$2^2 \cdot 3^2$
1	11	$2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	$2^5 \cdot 3 \cdot 17$
1	12	$2^{22} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	$2^{11} \cdot 3 \cdot 17$
2	12	$2^{14} \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4$	$2^3 \cdot 5$
2	13	$2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	$2^{10} \cdot 3^4$	$2^3 \cdot 5 \cdot 7 \cdot 17$
3	14	$2^{11} \cdot 3^4 \cdot 5 \cdot 7$	$2^{10} \cdot 3 \cdot 7$	$3^3 \cdot 5$
4	15	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	2

if  $(\alpha, k) = (0, 9)$  or  $(1, 10)$ . Thus, if  $\sigma$  is nontrivial then  $m = 1$  and  $(\alpha, k) = (0, 9)$  or  $(1, 10)$ . Moreover, in these cases, there is a unique  $\sigma_0 \in \mathfrak{S}_{\alpha,k}$  which acts as  $-1$  on  $G_{\alpha,k}$ . Since  $G_{0,9}$  is 2-elementary,  $-1$  acts trivially on it; this implies that  $\sigma_0$  is the identity when  $(\alpha, k) = (0, 9)$ . Since  $G_{1,10}$  is not 2-elementary, the action of  $-1$  on it is nontrivial, so that  $\sigma_0$  is nontrivial when  $(\alpha, k) = (1, 10)$ . We conclude that  $\mathcal{H}_{\alpha,k}$  is trivial when  $(\alpha, k) \neq (1, 10)$  and that  $\mathcal{H}_{1,10}$  has order 2.

To compute the degree of  $p_{\alpha,k}$ , suppose first that  $(\alpha, k) \neq (0, 9)$  or  $(1, 10)$ . Then  $\mathcal{I}_{\alpha,k} = \mathfrak{S}_{\alpha,k} / \mathcal{H}_{\alpha,k} = \mathfrak{S}_{\alpha,k}$ . Moreover, since  $G_{\alpha,k}$  is not 2-elementary,  $|O(G_{\alpha,k})/(\pm 1)| = |O(G_{\alpha,k})|/2$ . Thus, by Theorem (5.1),

$$\deg p_{\alpha,k} = \frac{|O(G_{\alpha,k})/(\pm 1)|}{|\mathcal{I}_{\alpha,k}|} = \frac{|O(G_{\alpha,k})|}{2|\mathfrak{S}_{\alpha,k}|}$$

If  $(\alpha, k) = (0, 9)$  then  $G_{0,9}$  is 2-elementary so that  $O(G_{0,9})/(\pm 1) = O(G_{0,9})$ . In addition,  $\mathcal{I}_{0,9} = \mathfrak{S}_{0,9} / \mathcal{H}_{0,9} = \mathfrak{S}_{0,9}$ , so that

$$\deg p_{0,9} = \frac{|O(G_{0,9})/(\pm 1)|}{|\mathcal{I}_{0,9}|} = \frac{|O(G_{0,9})|}{|\mathfrak{S}_{0,9}|}$$

Finally, if  $(\alpha, k) = (1, 10)$  then  $|O(G_{1,10})/(\pm 1)| = |O(G_{1,10})|/2$ . This time, however,  $|\mathcal{I}_{1,10}| = |\mathfrak{S}_{1,10} / \mathcal{H}_{1,10}| = |\mathfrak{S}_{1,10}|/2$ . Thus,

$$\deg p_{1,10} = \frac{|O(G_{1,10})/(\pm 1)|}{|\mathcal{I}_{1,10}|} = \frac{|O(G_{1,10})|}{|\mathfrak{S}_{1,10}|} \quad \text{Q.E.D.}$$

§ 6. Cremona transformations and the period maps

Let  $M_n = D_n^2/\Gamma_n$  and  $M_{\alpha,k} = D_{\alpha,k}/\Gamma_{\alpha,k}$ ; these are coarse moduli spaces which parametrize the projective equivalence classes of certain plane sextic curves with nodes or polarized K3 surfaces with rational double points (the polarization comes from the line bundle  $\mathcal{L}$ ). Since the period maps  $p_n$  and  $p_{\alpha,k}$  have finite degree greater than one, in contrast to the case of the usual global Torelli theorem for polarized K3 surfaces, the period point does not determine the projective equivalence class. A natural question arises: *What is determined by the period maps  $p_n$  and  $p_{\alpha,k}$ ?* The following theorem provides a partial answer to this question.

**Theorem (6.1).** *Let  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  be two polarized K3 surfaces coming from  $M_n$  ( $n \geq 2$ ) or  $M_{\alpha,k}$ , and let  $\mu: S \rightarrow X'$  and  $\mu': S' \rightarrow X'$  be the minimal desingularizations. If  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  give the same point in the period space, then there is an isomorphism  $\Phi: S \simeq S'$ . In particular, the surfaces  $X$  and  $X'$  are birationally isomorphic.*

*Proof.* Let  $M(X)$  and  $M(X')$  denote the natural sublattices of  $H^2(S, \mathbb{Z})$  and  $H^2(S', \mathbb{Z})$  respectively, which are isomorphic to one of the Cremona or Todorov lattices. Let  $N(X)$  and  $N(X')$  denote the orthogonal complements of  $M(X)$  and  $M(X')$  in  $H^2(X, \mathbb{Z})$  and  $H^2(X', \mathbb{Z})$ , respectively. By the definition of the period map, if  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  give the same point in the period space, there is an isometry  $\gamma: N(X) \rightarrow N(X')$  which preserves the Hodge structure and positive sign structure. We first claim that the isometry  $\gamma$  extends to an isometry  $\tilde{\gamma}: H^2(S, \mathbb{Z}) \rightarrow H^2(S', \mathbb{Z})$  which preserves the sign structures. Since  $H^2(S, \mathbb{Z})$  and  $H^2(S', \mathbb{Z})$  are isomorphic to the K3 lattice  $A$ , this follows from Theorem (3.2) (2) in the Cremona cases, and Theorem (3.3) in the Todorov cases. Now the extended isometry  $\tilde{\gamma}$  maps  $M(X)$  to  $M(X')$ ; since  $M(X)$  and  $M(X')$  are contained in  $NS(S)$  and  $NS(S')$  respectively,  $\tilde{\gamma}$  preserves the Hodge structures. But then by the weak global Torelli theorem for K3 surfaces [12],  $S$  and  $S'$  are isomorphic.

The second statement is an obvious consequence of the first. Q.E.D.

Note that this theorem does not guarantee that the isomorphism  $\Phi$  between  $S$  and  $S'$  preserves the polarization; typically,  $\Phi^*(\mu'^*(\mathcal{L}'))$  and  $\mu^*(\mathcal{L})$  are distinct line bundles on  $S$ .

Let  $(X, \mathcal{L})$  be a polarized K3 surface coming from  $M_n$  or  $M_{\alpha,k}$ , and let  $U \subset \mathbb{P}^N$  be the image of  $X$  under the morphism defined by the complete linear system  $|\mathcal{L}|$ . In the Cremona cases,  $U = \mathbb{P}^2$  and the map  $X \rightarrow U$  always has degree 2. In the Todorov cases, if  $(\alpha, k) \neq (0, 9)$  or  $(1, 10)$  and  $X$  is generic, then the map  $X \rightarrow U$  is an isomorphism; if  $(\alpha, k) = (0, 9)$  then  $U = \mathbb{P}^2$  and the map  $X \rightarrow U$  has degree 2; and if  $(\alpha, k) = (1, 10)$  then  $U = \bar{\mathbb{F}}^2$

is a quadric cone in  $\mathbf{P}^3$  and the map  $X \rightarrow U$  again has degree 2. (See [20; Lemma 5.4 and Corollary 7.6]; the fact that the image is a quadric cone when  $(\alpha, k) = (1, 10)$  is due to Catanese and Debarre [2].)

If  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  are two polarized K3 surfaces coming from  $M_n$  or  $M_{n,x}$  which are sent to the same point in period space, we have seen above that  $X$  and  $X'$  are birational. We now refine our previous question and ask: *Is the induced birational map  $U \dashrightarrow U'$  given by a Cremona transformation of the ambient projective space  $\mathbf{P}^N$ ?*

We address this question in two of our cases in the next two sections. We will in fact deal with a restricted version of the question, in which the only Cremona transformations considered are the “regular” Cremona transformations. We begin by reviewing the facts about these transformations, following Dolgachev [8]. Let  $\text{Cr}_N$  denote the group of all Cremona transformations of  $\mathbf{P}^N$ .

Let us choose projective coordinates  $Z_0, Z_1, \dots, Z_N$  on the source  $\mathbf{P}^N$  (resp.  $Z'_0, Z'_1, \dots, Z'_N$  on the target  $\mathbf{P}^N$ ) and let  $p_0, \dots, p_N$  be the coordinate vertices of  $\mathbf{P}^N$ . Consider the Cremona transformation  $T_{p_0 \dots p_N}$  defined by the following formula:

$$Z'_i = Z_0 \cdot Z_1 \cdots Z_{i-1} \cdot Z_{i+1} \cdots Z_N \quad \text{for } i = 0, 1, \dots, N.$$

A composition of such a transformation  $T_{p_0 \dots p_N}$  with a projective automorphism of  $\mathbf{P}^N$  is called a *standard* Cremona transformation of  $\mathbf{P}^N$ . Together with the projective automorphism group, the standard Cremona transformations generate a subgroup  $\text{Cr}_N^{\text{reg}}$  of  $\text{Cr}_N$ , called the group of *regular* Cremona transformations (cf. [3], [4], [8]). A regular Cremona transformation  $T: \mathbf{P}^N \dashrightarrow \mathbf{P}^N$  has a fundamental set which is completely determined by a finite set of points  $p_1, p_2, \dots, p_r$  (called the *fundamental points* of  $T$ ) in the sense that all other base conditions follow from assigning certain multiplicities at the given set of points. The inverse transformation  $T^{-1}$  has the same property, and it has the same number of fundamental points  $q_1, q_2, \dots, q_r$  as does  $T$ . Let

$$\lambda_1: V_1 = B(p_1, \dots, p_r) \dashrightarrow \mathbf{P}^N, \quad \lambda_2: V_2 = B(q_1, \dots, q_r) \dashrightarrow \mathbf{P}^N$$

be the blow ups of the fundamental points of  $T$  and  $T^{-1}$  respectively. Then there exists a pair of birational morphisms  $\nu_1: V \rightarrow V_1$  and  $\nu_2: V \rightarrow V_2$  such that as birational mappings,  $T \circ \lambda_1 \circ \nu_1 = \lambda_2 \circ \nu_2$ .

$$\begin{array}{ccc} V_1 & \xleftarrow{\nu_1} & V & \xrightarrow{\nu_2} & V_2 \\ \lambda_1 \downarrow & & & & \downarrow \lambda_2 \\ \mathbf{P}^N & \dashrightarrow & & & \mathbf{P}^N \end{array}$$

In the case that  $N=2$ , Max Noether's theorem on the decomposition of birational automorphisms of  $P^2$  says that  $Cr_2 = Cr_2^{reg}$ , that is, all Cremona transformations of  $P^2$  are regular. However, if  $N$  is greater than 2, it is well known that  $Cr_N$  is a very big group which contains  $Cr_N^{reg}$  as a proper subgroup.

§7. Cremona equivalence of irreducible sextic curves with nodes

Let  $C$  be an irreducible sextic curve in  $P^2$  with  $n$  nodes  $p_1, \dots, p_n$ . Consider the standard Cremona transformation  $T_{p_1 p_2 p_3}: P^2 \dashrightarrow P^2$  whose fundamental points are the first three points  $p_1, p_2, p_3$ . The transformation  $T_{p_1 p_2 p_3}$  is nothing but the usual quadratic transformation and it is easy to see that it maps the curve  $C$  to another irreducible sextic  $C'$  with  $n$  nodes.<sup>5</sup> (See Figure 1.) Since an arbitrary Cremona transformation  $T \in Cr_2$  can be expressed as a composite of standard Cremona transformations,  $T$  has the same property: it maps any sextic curve with nodes at its fundamental points to a sextic curve with nodes. We call two irreducible sextics  $C$  and  $C'$  Cremona equivalent if there exists a Cremona transformation  $T \in Cr_2$  whose fundamental points are contained among the nodes of  $C$ , such that  $T(C) = C'$ .

Let  $M_n^{Cr}$  denote the set of Cremona equivalence classes of irreducible sextics with  $n$  nodes for  $2 \leq n \leq 10$ . Recall that  $M_n$  is the moduli space of projective equivalence classes of irreducible sextics with  $n$  nodes. There is a natural map  $Cr: M_n \rightarrow M_n^{Cr}$  which associates the Cremona equivalence class of  $C$  to the projective equivalence class of  $C$ .

The following theorem provides a positive answer to the question asked at the end of Eection 6 in this case.

**Theorem (7.1).** *Let  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  be two polarized K3 surfaces in  $M_n$  ( $n \geq 2$ ). Let  $\mu: S \rightarrow X$  and  $\mu': S' \rightarrow X'$  be the minimal desingularizations and let  $C$  and  $C'$  be the associated irreducible sextics with  $n$  nodes. Assume that the Picard numbers of  $S$  and  $S'$  are both equal to  $n+1$ . Then  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  have the same period in  $D_n^\circ / Aut_2(D_n)$  if and only if  $C$  and  $C'$  are Cremona equivalent.*

Let  $M_n^\circ$  denote the subset of  $M_n$  which consists of polarized K3 surfaces whose minimal desingularizations have Picard numbers  $n+1$  and let  $M_n^{Cr}$

<sup>5</sup> Here we implicitly assume that  $C$  is a "general" sextic with  $n$  nodes. The meaning of "general sextic with  $n$  nodes" is as follows: the line  $\overline{p_i p_j}$  ( $i \neq j$ ) through the nodes  $p_i$  and  $p_j$  of  $C$  intersects  $C$  transversely at the two residual points in  $C \cap \overline{p_i p_j}$  for all pairs  $(i, j)$ . If the minimal resolution  $S$  of the associated K3 surface  $X$  in  $M_n$  has Picard number  $n+1$ , the curve  $C$  is general.

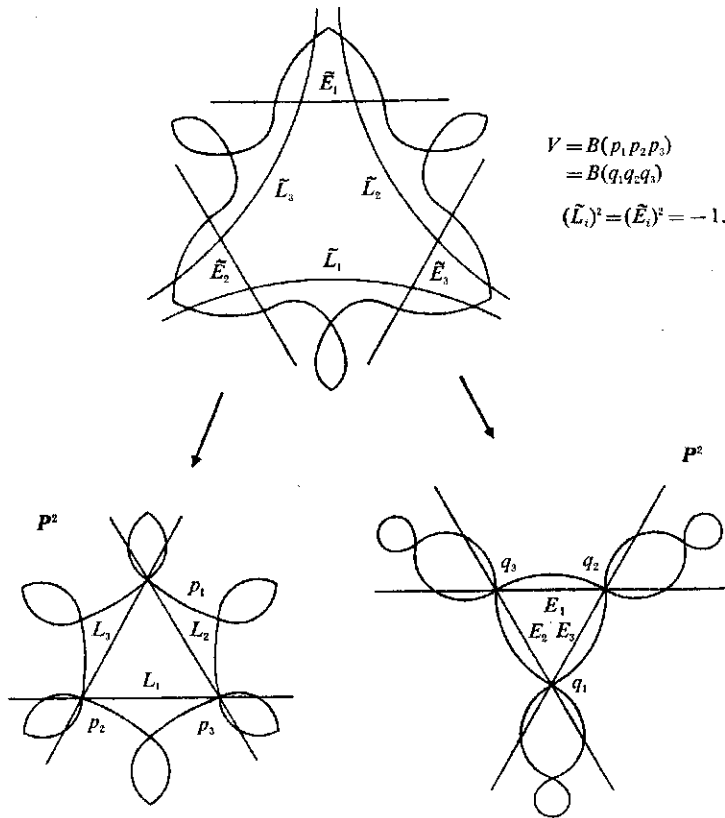


Figure 1.

denote the image of  $M_n^\circ$  under the map Cr. Then  $M_n^\circ$  (resp.  $M_n^{\circ\text{Cr}}$ ) is an open dense subset in  $M_n$  (resp.  $M_n^{\text{Cr}}$ ). Similarly, we can define an open dense subset  $D_n^{\circ*}$  of the period space  $D_n^\circ$  such that  $D_n^{\circ*}/\Gamma_n$  is isomorphic to  $M_n^\circ$ .

Then from Theorem (7.1), we get:

**Corollary (7.2).** *The set  $M_n^{\circ\text{Cr}}$  is isomorphic to  $D_n^{\circ*}/\text{Aut}_Z(D_n)$  and the period map  $p_n: M_n^\circ \rightarrow D_n^{\circ*}/\text{Aut}_Z(D_n)$  can be naturally identified with the map Cr:  $M_n^\circ \rightarrow M_n^{\circ\text{Cr}}$ .*

*Proof of Theorem (7.1).* If  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  have the same period, by Theorem (6.1) there exists an isomorphism  $\Phi: S \simeq S'$ . Since we assume that the Picard numbers of  $S$  and  $S'$  are both equal to  $n+1$ , we have  $\text{NS}(S) = M(X)$  and  $\text{NS}(S') = M(X')$ ; in this case,  $\Phi$  induces isomorphisms

$\Phi^*: M(X') \simeq M(X)$  and  $\Phi^*: N(X') \simeq N(X)$ . Let  $p_1, \dots, p_n$  (resp.  $q_1, \dots, q_n$ ) be the nodes of  $C$  (resp.  $C'$ ) and let  $\lambda: V = B(p_1, \dots, p_n) \rightarrow \mathbf{P}^2$  (resp.  $\lambda': V' = B(q_1, \dots, q_n) \rightarrow \mathbf{P}^2$ ) be the blow up of  $\mathbf{P}^2$  at  $p_1, \dots, p_n$  (resp.  $q_1, \dots, q_n$ ). We then get finite maps  $g: S \rightarrow V$  and  $g': S' \rightarrow V'$  of degree 2; let  $\iota: S \rightarrow S$  and  $\iota': S' \rightarrow S'$  be the corresponding involutions. Since  $g^*H^2(V, \mathbf{Q}) = M(X) \otimes \mathbf{Q}$ ,  $M(X)$  is the invariant part of the cohomology  $H^2(S, \mathbf{Z})$  under the action of  $\iota^*$ ; a similar statement holds for  $S'$ . Thus,  $\Phi^*$  maps the invariant part of  $H^2(S', \mathbf{Z})$  to the invariant part of  $H^2(S, \mathbf{Z})$ , so that the isomorphism  $\Phi$  descends to an isomorphism  $\bar{\Phi}: V \simeq V'$ , which maps the proper transform of  $C$  to the proper transform of  $C'$ . Hence  $C$  and  $C'$  are Cremona equivalent. The converse statement in the theorem is obvious.

Q.E.D.

**Remark (7.3).** A K3 surface  $X$  of Todorov type  $(0, 9)$  is the double cover of  $\mathbf{P}^2$  branched along two cubics  $D_1$  and  $D_2$  intersecting transversely (when  $X$  is general in moduli). Under the Cremona transformations whose fundamental points are contained in  $D_1 \cap D_2$ ,  $D_1$  and  $D_2$  are mapped to cubics  $D'_1$  and  $D'_2$ . Let  $M_{0,9}^{Cr}$  denote the Cremona equivalence classes of pairs of cubics. Then as in Theorem (7.1) and Corollary (7.2), the natural morphism  $Cr: M_{0,9}^o \rightarrow M_{0,9}^{Cr}$  can be identified with the period map  $p_{0,9}: M_{0,9}^o \rightarrow D_{0,9}/Aut_{\mathbf{Z}}(D_{0,9})$ . Here we denote by  $M_{0,9}^o$  the moduli space of K3 surfaces of Todorov type  $(0, 9)$  whose minimal desingularizations have Picard numbers 10.

We now explore the relationship between the degrees of the period maps and ‘‘Coble’s numbers’’. Let  $P_n^2$  denote the set of projective equivalence classes of unordered  $n$ -tuples of points in  $\mathbf{P}^2$ . The Cremona group  $Cr_2$  acts on  $P_n^2$  in a natural manner (cf. [3], [8]). Let  $p_1, \dots, p_n$  denote a set of  $n$  points in  $\mathbf{P}^2$ . We let  $Cob(p_1, \dots, p_n)$  denote the number of projective equivalence classes which are Cremona equivalent to the given set  $(p_1, \dots, p_n)$  and call this the *Coble number* of the set of points.

The following theorem is due to Coble [3], [5].

**Theorem (7.4).** (1) For  $5 \leq n \leq 10$ , let  $p_1, \dots, p_n$  be the  $n$  nodes of a general irreducible sextic curve  $C$  with  $n$  nodes in  $\mathbf{P}^2$ . Then the Coble number  $Cob = Cob(p_1, \dots, p_n)$  is given as follows:

$n$	5	6	7	8	9	10
Cob	1	72	288	8640	$2^8 \cdot 960$	$2^{13} \cdot 31 \cdot 51$

(2) If  $p_1, \dots, p_9$  are the base points of a pencil of cubics, then the Coble number  $Cob(p_1, \dots, p_9)$  is equal to 960.

Note that for  $n \leq 8$ , there exists a sextic curve  $C$  with nodes at any specified set of  $n$  points  $p_1, \dots, p_n$ ; for  $n=9$  and  $10$  the sets of points we are considering are rather special.

Comparing Table 6 and the Coble numbers, one would expect to find some relations between the degrees of the period maps  $p_n$  and  $p_{0,9}$  and the Coble numbers. In fact, we can reprove this theorem of Coble by using our computations of the degrees of these period maps.

Let  $\psi_n: M_n \rightarrow P_n^2$  denote the natural map which associates to the projective equivalence class of the nodal sextic  $C$  the projective equivalence class of its set of nodes. If  $n$  is less than or equal to  $8$ , the map  $\psi_n$  is generically surjective, and if  $n$  is greater than  $8$ , the image of  $\psi_n$  is a proper subset of  $P_n^2$ . Let  $P_n^{2, Cr}$  denote the set of Cremona equivalence classes of unordered  $n$ -tuples of points in  $P^2$  and let  $W_n: P_n^2 \rightarrow P_n^{2, Cr}$  be the natural map. We have the commutative diagram

$$\begin{array}{ccc} M_n & \xrightarrow{\psi_n} & P_n^2 \\ \text{Cr} \downarrow & & \downarrow W_n \\ M_n^{Cr} & \longrightarrow & P_n^{2, Cr} \end{array}$$

Let  $\lambda: V \rightarrow P^2$  be the blow up of  $n$  points  $p_1, \dots, p_n$  which are either the  $n$  nodes of an irreducible sextic curve  $C$  or the  $9$  base points of a pencil of cubic curves  $|D|$ . (We refer to the latter situation as the “(0, 9) case”.) Consider the linear system  $|-2K_V|$ , which contains the proper transform of  $C$  on  $V$  (or the proper transform of  $D_1 + D_2$  for  $D_i \in |D|$  in the (0, 9) case). One can easily see the following lemmas (cf. [8]):

**Lemma (7.5).** *Let  $\text{Aut}(V)$  denote the automorphism group of  $V$ . If  $C$  (or  $|D|$ ) is generic, we have the following:*

(1) *If  $n=5$ ,  $\text{Aut}(V) \cong (\mathbb{Z}/2\mathbb{Z})^4$  and all non-trivial elements in  $\text{Aut}(V)$  act on  $|-2K_V|$  non-trivially.*

(2) *If  $n=6$ ,  $\text{Aut}(V) = \{1\}$ .*

(3) *If  $n=7$ ,  $\text{Aut}(V) = \mathbb{Z}/2\mathbb{Z}$  and it acts non-trivially on  $|-2K_V|$ .*

(4) *If  $n=8, 9$ , or  $10$ , all elements of  $\text{Aut}(V)$  act trivially on  $|-2K_V|$ .*

(5) *In the (0, 9) case, all elements of  $\text{Aut}(V)$  act trivially on  $|-2K_V|$ .*

*Proof.* A proof of this lemma for  $n \leq 8$  can be found in Dolgachev [8]; we will treat the (0, 9) case and the cases  $n=9$  and  $n=10$ .

In the (0, 9) case, the pencil of cubics  $|D|$  lifts to an elliptic pencil  $|-K_V|$  on  $V$  which has a section. Moreover,  $\text{Sym}^2 H^0(-K_V) \cong H^0(-2K_V)$ , so it suffices to show that  $\text{Aut}(V)$  acts trivially on  $|-K_V| \cong P^1$ . If  $\gamma \in \text{Aut}(V)$  acts nontrivially, by composing with a translation along the fibers we may assume that  $\gamma$  preserves some specified section of the pencil.

Suppose that  $|D|$  is generic. It must then be a Lefschetz pencil so that the elliptic pencil  $|-K_V|$  together with a choice of section gives  $V$  the structure of a (rational) Weierstrass fibration over  $P^1$ , and this structure is preserved by the action of  $\gamma$ . A moduli space for such fibrations has been constructed by Miranda [16] (using geometric invariant theory) from their description in terms of a pair of sections  $g_4 \in \text{Sym}^4 W$ ,  $g_6 \in \text{Sym}^6 W$ , where  $W = H^0(P^1, \mathcal{O}_{P^1}(1))$ . To construct the moduli space, one must form the quotient of the space  $U = \text{Sym}^4 W \oplus \text{Sym}^6 W$  by the group  $G = \text{SL}(W) \times C^*$ , which acts via the natural actions of  $\text{SL}(W)$  in the first factor, and the action  $\lambda \rightarrow (\lambda^4, \lambda^6)$  in the second factor; Miranda shows that the set of stable points for this action is nonempty. In particular, the automorphisms of the generic rational Weierstrass fibration are given by the elements of  $G$  which act trivially on  $U$ . But from the standard representation theory of  $\text{SL}_2$  it follows that the only elements of  $G$  acting trivially on  $U$  are those of the form  $(\pm 1) \times (\pm 1)$ ; since these elements act trivially on  $P^1$  as well, we conclude that the generic rational Weierstrass fibration has no automorphism acting nontrivially on  $P^1$ . This means that  $\gamma$  must act trivially on  $P^1$  when  $|D|$  is generic, proving (5).

If  $n=9$ ,  $|-2K_V|$  is an elliptic pencil with one double fiber; by a result of Dolgachev [7] (cf. also [23; pp. 22–24]),  $V$  is the logarithmic transform of a Weierstrass fibration  $f: W \rightarrow P^1$ . Conversely, by [23; Proposition 1.5] the logarithmic transform of order 2 along a single smooth fiber of a generic Weierstrass fibration has a minimal model isomorphic to  $P^2$ , and so must coincide with the blow up of the nodes of some sextic curve in  $P^2$ . Our results in the  $(0, 9)$  case now imply that when  $W$  is generic, the set of singular fibers of  $f$  is not invariant under any nontrivial automorphism of  $P^1$ . Since the set of non-multiple singular fibers is preserved under logarithmic transform,  $V$  has the same property; this implies that every automorphism of  $V$  acts trivially on  $|-2K_V| \cong P^1$ .

Finally, if  $n=10$  and  $V$  is generic, the linear system  $|-2K_V|$  has (projective) dimension zero, so any element of  $\text{Aut}(V)$  acts trivially on  $|-2K_V|$ .  
 Q.E.D.

For an automorphism  $g$  of  $V$ , we define a Cremona transformation  $h_g = \lambda \circ g \circ \lambda^{-1}$  which makes the following diagram commutative:

$$\begin{array}{ccc}
 V & \xrightarrow{g} & V \\
 \lambda \downarrow & & \downarrow \lambda \\
 P^2 & \xrightarrow{h_g} & P^2
 \end{array}$$

Let  $C \in M_n$  denote an irreducible sextic curve with  $n$  nodes at the given

points  $p_1, \dots, p_n$  and let  $h_g(C)$  be the image of  $C$  under the Cremona transformation  $h_g$ . Then we have the following:

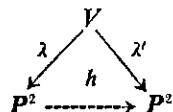
**Lemma (7.6).** *Let  $n$  be an integer such that  $4 \leq n \leq 10$ .*

(1) *For any element  $g$  in  $\text{Aut}(V)$ , the curve  $h_g(C)$  is an irreducible sextic with  $n$  nodes at the same set of points  $p_1, \dots, p_n$ . In particular, we have  $\text{Cr}(C) = \text{Cr}(h_g(C))$  and  $\psi_n(C) = \psi_n(h_g(C))$ .*

(2) *Assume that all non-trivial elements in  $\text{Aut}(V)$  act on  $|-2K_V|$  nontrivially and  $C$  is general. Then for a non-trivial element  $g$  in  $\text{Aut}(V)$ , the curve  $h_g(C)$  is not projectively equivalent to the original curve  $C$ .*

(3) *Conversely, let  $C'$  be an irreducible sextic with  $n$  nodes at the same set of points  $p_1, \dots, p_n$  which is Cremona equivalent to  $C$ . Then there exists an element  $g$  in  $\text{Aut}(V)$  such that the Cremona transformation  $h_g = \lambda \circ g \circ \lambda^{-1}$  maps  $C$  to  $C'$ .*

*Proof.* The assertion (1) follows directly from the definition of the Cremona transformation  $h_g$ . To prove (2), let  $\bar{C}$  be the proper transform of  $C$  by  $\lambda$ . Then we have  $\bar{C} \in |-2K_V|$ . The assumption means that for a non-trivial element  $g$  in  $\text{Aut}(V)$ ,  $g(\bar{C}) \neq \bar{C}$ . This also implies that  $h_g(C) \neq C$ . If we take the set of  $n$  points  $p_1, \dots, p_n$  in general position (this is possible since  $C$  is general), there are no non-trivial elements in  $\text{Aut}(\mathbb{P}^2)$  which stabilize the set of  $n$  nodes  $p_1, \dots, p_n$  if  $n \geq 4$ . Since the set of  $n$  nodes of  $h_g(C)$  coincides with that of  $C$ , (i.e.,  $p_1, \dots, p_n$ ), this implies that  $h_g(C)$  is not projectively equivalent to  $C$ . (3) By assumption, there exists a Cremona transformation  $h: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  with base points  $p_1, \dots, p_n$  which maps  $C$  to  $C'$ . Then there exists a birational morphism  $\lambda': V \rightarrow \mathbb{P}^2$  which makes the following diagram commutative:



Since  $h$  maps  $C$  to  $C'$ ,  $h$  also maps the set of  $n$  points  $p_1, \dots, p_n$  to the same set. Hence  $\lambda'$  is also the blow up of  $n$  points  $p_1, \dots, p_n$ . By the universality of the blow up, there exists an automorphism  $g: V \rightarrow V$  such that  $\lambda' = \lambda \circ g$ . Thus we have  $h = \lambda \circ g \circ \lambda^{-1}$  as desired. Q.E.D.

From these lemmas and the commutative diagram above, we have the following

**Proposition (7.7).** *Let  $W_n: \mathbb{P}_n^2 \rightarrow \mathbb{P}_n^{2, \text{Cr}}$  be the natural map so that the degree of  $W_n$  is equal to the Coble number.*

- (1) If  $n=5$ ,  $\deg W_n = (\deg p_n)/16$ .
- (2) If  $n=6$ ,  $\deg W_n = \deg p_n$ .
- (3) If  $n=7$ ,  $\deg W_n = (\deg p_n)/2$ .
- (4) If  $n=8$ ,  $\deg W_n = \deg p_n$ .
- (5) If  $n=9$  or  $10$ , on the image of  $\psi_n$ ,  $\deg W_n = \deg p_n$ .
- (6) In the  $(0, 9)$  case, on the image of the natural map  $\psi_{0,9}: M_{0,9} \rightarrow P^2$ ,  $\deg W_9 = \deg p_{0,9}$ .

Proposition (7.7) and Tables 6 and 8 now reprove Coble's Theorem (7.4).

### § 8. The Cayley Symmetroid

In this section, we study K3 surfaces of Todorov type  $(0, 10)$ : these are related to quartic K3 surfaces with 10 nodes in  $P^3$ . By using the period map of these K3 surfaces, we will find an interpretation of Coble's classical results [4] on the Cayley symmetroids. Coble's results have recently been given another modern interpretation by Cossec and Dolgachev [6]; our treatment of this topic depends heavily on their work, and will be relatively brief.

Let  $(X, \mathcal{L}, \Sigma)$  be a (polarized) K3 surface of Todorov type  $(0, 10)$ . We call  $(X, \mathcal{L}, \Sigma)$  simple if the linear system  $|\mathcal{L}|$  is not hyperelliptic, and if there are exactly 10 irreducible rational curves  $C_1, \dots, C_{10}$ , on  $S$  which are orthogonal to  $\mu^*\mathcal{L}$ . (The generic K3 surface of Todorov type  $(0, 10)$  is simple by [20; Corollary 7.6]). For a simple surface, the distinguished partial desingularization  $\Sigma$  of  $X$  actually coincides with  $X$ , so that the pair  $(X, \mathcal{L})$  contains all the information we need to discuss the moduli.  $X$  is embedded by  $|\mathcal{L}|$  as a quartic hypersurface with exactly 10 ordinary double points; such a quartic hypersurface is called a *Cayley symmetroid*.

The original definition of a Cayley symmetroid was as follows (cf. [4]): take a  $4 \times 4$  symmetric matrix  $A$  whose entries are linear forms on  $P^3$ . The quartic hypersurface  $X$  defined by the equation  $\det(A)=0$  is then a Cayley symmetroid. But one easily checks that on the minimal desingularization  $\mu: S \rightarrow X$ ,  $(1/2)(c_1(\mu^*(\mathcal{L})) + \sum c(C_i)) \in H^2(S, \mathbf{Z})$ . Hence,  $(X, \mathcal{L}, X)$  is a K3 surface of Todorov type  $(0, 10)$  as well.

**Lemma (8.1).** *If  $X$  is a Cayley symmetroid whose minimal desingularization has Picard number 11, then no four of the nodes of  $X$  are coplanar.*

*Proof.* Assume that four nodes  $p_1, p_2, p_3, p_4$  of  $X$  are coplanar. Projecting from the point  $p_1$  to a complementary  $P^2$ , we get a morphism  $\psi: B_{p_1}(X) \rightarrow P^2$  of degree two, where  $B_{p_1}(X)$  denotes the blow up of  $X$  at  $p_1$ . The branch divisor of this morphism is a plane sextic curve with 9 nodes.

Since the four nodes on  $X$  are coplanar, three of the nodes in the branch curve must be collinear. If  $l$  denotes a line containing three nodes, then  $l$  meets the branch curve *only* in those three nodes so that the inverse image of  $l$  in  $B_{p_1}(X)$  has two components  $l_1$  and  $l_2$ .  $l_1 - l_2$  then lifts to a divisor class on the minimal desingularization  $\mu: S \rightarrow X$  whose first Chern class is not in the linear span of  $c_1(\mu^*\mathcal{L})$  and the exceptional classes  $c_1(C_i)$ , so that  $S$  has Picard number greater than 11, contrary to hypothesis. Q.E.D.

Let  $X$  be a Cayley symmetroid with ten nodes  $p_1, \dots, p_{10}$  and let  $\mu: S \rightarrow X$  be the minimal desingularization; suppose that the Picard number of  $S$  is 11. Let  $\mathcal{L} = \mu^*\mathcal{O}_{\mathbb{P}^3}(1)$  and  $C_i = \mu^{-1}(p_i)$ . Choose four of the points  $p_1, p_2, p_3, p_4$ ; since these are not coplanar, we can define the *standard Cremona transformation*  $T_{p_1 p_2 p_3 p_4}: \mathbb{P}^3 \rightarrow \mathbb{P}^3$  with fundamental points  $p_1, p_2, p_3, p_4$  (as in Section 6). It is easy to see that the image  $X' \subset \mathbb{P}^3$  of  $X$  under  $T_{p_1 p_2 p_3 p_4}$  is also a symmetroid and that there is a morphism  $\mu': S \rightarrow X'$ . Let  $q_1, \dots, q_{10}$  denote the nodes of  $X'$  and put  $\mathcal{L}' = \mu'^*(\mathcal{O}_{\mathbb{P}^3}(1))$  and  $C'_i = (\mu')^{-1}(q_i)$ . For a suitable numbering of the  $q_i$ 's, we then have the following relations in the Picard group of  $S$ :

$$\begin{aligned} \mathcal{L}' &= \mathcal{L}^{\otimes 8} \otimes \mathcal{O}_S(-2C_1 - 2C_2 - 2C_3 - 2C_4) \\ \mathcal{O}_S(C'_i) &= \mathcal{L} \otimes \mathcal{O}_S(-C_1 - C_2 - C_3 - C_4 + C_i) && \text{for } 1 \leq i \leq 4 \\ \mathcal{O}_S(C'_i) &= \mathcal{O}_S(C_i) && \text{for } 5 \leq i \leq 10. \end{aligned}$$

We shall now relate the action of regular Cremona transformations in  $Cr_3^{\text{reg}}$  on the Todorov lattice  $M_{0,10}$  to the action of the *Weyl group*  $W_{2,4,6}$  associated to the Dynkin diagram  $T_{2,4,6}$  (cf. [8], [6]). Fix an isomorphism  $\alpha: \{1, \dots, 10\} \rightarrow \text{Sing } \Sigma_{0,10}$ , and let  $e_i$  denote  $e_{\alpha(i)}$ . Define the "roots"  $\beta_i$  in  $M_{0,10}$  as follows:

$$\begin{aligned} \beta_0 &= \lambda - e_1 - e_2 - e_3 - e_4 \\ \beta_i &= e_i - e_{i+1} && \text{for } 1 \leq i \leq 9. \end{aligned}$$

If we define a new bilinear form on  $M_{0,10}$  by  $(x, y) = -(x \cdot y)/2$ , then  $(\beta_i, \beta_i) = 2$ ; the Dynkin diagram of the (generalized) root system spanned by  $\beta_0, \beta_1, \dots, \beta_9$  (with respect to the bilinear form  $(,)$ ) is given by the diagram  $T_{2,4,6}$  in Figure 2.

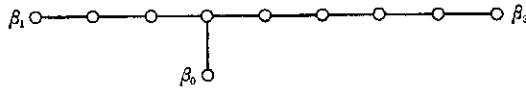


Figure 2.

Define the simple reflection  $s_j$ ,  $0 \leq j \leq 9$  on  $M_{0,10}$  as follows:

$$s_j(x) = x - (\beta_j, x)\beta_j.$$

It is easy to check that  $s_j$  defines an isometry of  $M_{0,10}$ ; the Weyl group  $W_{2,4,6}$  is the group of isometries of  $M_{0,10}$  generated by these simple reflections  $s_j$ ,  $j=0, \dots, 9$ .

The simple reflection  $s_j$ ,  $j=1, \dots, 9$  acts on  $M_{0,10}$  by leaving  $e_i$  fixed for  $i \neq j, j+1$ , and interchanging  $e_j$  and  $e_{j+1}$ . The simple reflection  $s_0$  acts on  $M_{0,10}$  by the following formula:

$$\begin{aligned} s_0(\lambda) &= 3\lambda - 2(e_1 + e_2 + e_3 + e_4) \\ s_0(e_i) &= \lambda - (e_1 + e_2 + e_3 + e_4) + e_i \quad \text{for } 1 \leq i \leq 4 \\ s_0(e_i) &= e_i \quad \text{for } 5 \leq i \leq 10. \end{aligned}$$

Let  $\alpha_j = \alpha \circ (j, j+1) \circ \alpha^{-1} \in \text{Aut}(\text{Sing } \Sigma_{0,10})$ , where  $(j, j+1)$  denotes a simple transposition.

**Proposition (8.2).** *Let  $(X, \mathcal{L})$  be a (simple) K3 surface of Todorov type  $(0, 10)$ , let  $\mu: S \rightarrow X$  be the minimal desingularization, let  $\phi: H^2(S, \mathbb{Z}) \rightarrow \Lambda$ ,  $\psi: \text{Sing } X \rightarrow \text{Sing } \Sigma_{0,10}$  be a special marking of  $X$ , and suppose that  $S$  has Picard number 11.*

- (1) *For  $1 \leq j \leq 9$ ,  $(s_j \circ \phi, \alpha_j \circ \psi)$  is a special marking of  $X$ .*
- (2) *Let  $X'$  be the image of  $X$  under the standard Cremona transformation  $T_{p_1 p_2 p_3 p_4}$ . Then  $(s_0 \circ \phi, \psi)$  is a special marking of  $X'$ .*
- (3) *If  $w \in W_{2,4,6}$  there exists a regular Cremona transformation  $T$  whose fundamental points are contained in  $\text{Sing } X$  and a permutation  $\sigma \in \mathfrak{S}_{0,10}$  such that  $(w \circ \phi, \sigma \circ \psi)$  is a special marking of the image  $X'$  of  $X$  under  $T$ .*
- (4) *If  $T$  is a regular Cremona transformation whose fundamental points are contained in  $\text{Sing } X$ , and  $X'$  is the image of  $X$  under  $T$  then there exist  $w \in W_{2,4,6}$  and  $\sigma \in \mathfrak{S}_{0,10}$  such that  $(w \circ \phi, \sigma \circ \psi)$  is a special marking of  $X'$ .*

*Proof.* (1) follows directly from the definitions of special marking. (2) is a consequence of the formulas we have given above for the actions of  $s_0$  on  $M_{0,10}$  and of  $T_{p_1 p_2 p_3 p_4}$  on the Picard group of  $S$ . (3) follows from (1) and (2), by the definition of  $W_{2,4,6}$ . For (4), it is enough to prove the statement when  $T$  is a standard Cremona transformation whose fundamental points are contained in  $\text{Sing } X$ . But any such transformation can be expressed as the composite of a permutation of  $\text{Sing } X$  and the transformation  $T_{p_1 p_2 p_3 p_4}$ ; the statement now follows from (1) and (2). Q.E.D.

The reflections  $s_j$  all lie in  $O_-(M_{0,10})$ , so there is a natural homomorphism  $W_{2,4,6} \rightarrow O_-(M_{0,10})$ . We let  $f: W_{2,4,6} \rightarrow O(G_{0,10})/(\pm 1)$  be the composite

of this homomorphism with the canonical homomorphism  $O_-(M_{0,10}) \rightarrow O(G_{0,10})/(\pm 1)$ . The following lemma is originally due to Coble [4] (a recent modern proof has been given by Cossec and Dolgachev [6]):

**Lemma (8.3).**  $W_{2,4,6}$  contains three distinguished elements  $K, B,$  and  $G$  (called the Kantor, dilated Bertini, and dilated Geiser involutions) such that the smallest normal subgroup  $\bar{W}(2)$  of  $W_{2,4,6}$  containing  $K, B,$  and  $G$  satisfies  $W_{2,4,6}/\bar{W}(2) \cong Sp(8, 2)$ .

We use Lemma (8.3) to deduce

**Lemma (8.4).** The homomorphism  $f: W_{2,4,6} \rightarrow O(G_{0,10})/(\pm 1)$  is surjective.

*Proof.* By using the explicit generators of  $G_{0,10}$  given in [20; Proposition 6.2], one can check that the homomorphism  $f$  is non-trivial but that  $K, B,$  and  $G$  (the involutions from Lemma (8.3)) all lie in  $\text{Ker } f$ ; by Lemma (8.3),  $\bar{W}(2)$  is contained in  $\text{Ker } f$ . Since  $W_{2,4,6}/\bar{W}(2) \cong Sp(8, 2)$  is a simple group, we have  $\bar{W}(2) = \text{Ker } f$  and  $f(W_{2,4,6}) \cong Sp(8, 2)$ . On the other hand, by Corollary (2.4), there is an exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow O(G_{0,10}) \longrightarrow Sp(8, 2) \longrightarrow 1$$

and it is easy to see that the  $\mathbb{Z}/2\mathbb{Z}$  subgroup (which comes from the  $w_{2,2}^3$  summand in the normal form decomposition of  $G_{0,10}$ ) is generated by  $-1$ . Thus,  $O(G_{0,10})/(\pm 1) \cong Sp(8, 2)$ , so that  $f$  must be surjective. Q.E.D.

We can now give an answer to the question posed at the end of Section 6 in the Todorov case (0, 10).

**Theorem (8.5).** Let  $(X_1, \mathcal{L}_1)$  and  $(X_2, \mathcal{L}_2)$  be two K3 surfaces of Todorov type (0, 10), regarded as Cayley symmetroids in  $\mathbb{P}^3$ . Suppose that the minimal desingularizations of  $X_1$  and  $X_2$  have Picard number 11. Then  $(X_1, \mathcal{L}_1)$  and  $(X_2, \mathcal{L}_2)$  have the same period in  $D_{0,10}/\text{Aut}_{\mathbb{Z}}(D_{0,10})$  if and only if there exists a regular Cremona transformation  $T \in \text{Cr}_3^{\text{reg}}$  such that  $T(X_1, \mathcal{L}_1) = (X_2, \mathcal{L}_2)$ .

*Proof.* Suppose that  $(X_1, \mathcal{L}_1)$  and  $(X_2, \mathcal{L}_2)$  have the same period. By Theorem (6.1), the minimal desingularizations of  $X_1$  and  $X_2$  are isomorphic; we use the isomorphism to identify them with a single smooth K3 surface  $S$ , with  $\mu_i: S \rightarrow X_i$  the desingularization maps for  $i = 1, 2$ .

Let  $\phi_i: H^2(S, \mathbb{Z}) \rightarrow \mathbb{A}$ ,  $\psi_i: \text{Sing } X \rightarrow \text{Sing } \Sigma_{0,10}$  be special markings of  $X_i$  for  $i = 1, 2$ . By Lemma (8.4), there is some  $w \in W_{2,4,6}$  which has the same image in  $O(G_{0,10})/(\pm 1)$  as does  $(\phi_2 \circ \phi_1^{-1})|_{M_{0,10}}$ . By Proposition (8.2) (3), there is a regular Cremona transformation  $T$  and a permutation  $\sigma \in \mathfrak{S}_{0,10}$

such that if  $(X_3, \mathcal{L}_3) = T(X_1, \mathcal{L}_1)$  then  $(w \circ \phi_1, \sigma \circ \psi_1)$  is a special marking of  $X_3$ .

By construction,  $(\phi_2 \circ (w \circ \phi_1)^{-1})|_{M_{0,10}}$  maps to  $(-1)^k$  in  $O(G_{0,10})$  for some  $k$ ; let  $\tau = (1_{M_{0,10}}) \oplus (-1)^k (\phi_2 \circ (w \circ \phi_1)^{-1})|_{N_{0,10}}$ . Then  $\tau \in O_-(A)$ , so that if we define  $\phi_3 = \tau \circ w \circ \phi_1$  and  $\psi_3 = \sigma \circ \psi_1$ , then  $(\phi_3, \psi_3)$  is also a special marking of  $X_3$ . But since  $(\phi_3 \circ \phi_2^{-1})|_{N_{0,10}} = (-1_{N_{0,10}})^k$ ,  $\phi_2$  and  $\phi_3$  correspond to the same point of  $D_{0,10}/\Gamma_{0,10}$ ; hence  $(X_3, \mathcal{L}_3) = T(X_1, \mathcal{L}_1)$  is isomorphic to  $(X_2, \mathcal{L}_2)$ .

The converse statement is an obvious consequence of Proposition (8.2)

(4). Q.E.D.

Let  $M_{0,10}^{Cr}$  denote the moduli of regular Cremona equivalence classes of Cayley symmetroids  $X \subset P^3$ , and let  $Cr: M_{0,10} \rightarrow M_{0,10}^{Cr}$  be the natural map (assigning the Cremona equivalence class to the projective equivalence class). From Theorem (8.5) and Table 8 in Section 5, we have the following corollaries which explain the relation between Cremona transformations and the period map for K3 surfaces of Todorov type (0, 10).

**Corollary (8.6).** *Let  $M_{0,10}^\circ$  denote the moduli of the Cayley symmetroids whose minimal resolution have Picard number 11, and let*

$$p_{0,10}: M_{0,10} \longrightarrow D_{0,10}/\text{Aut}_{\mathbb{Z}}(D_{0,10})$$

*be the period map. Then the image of  $M_{0,10}^\circ$  under  $p_{0,10}$  and under the map  $Cr$  are isomorphic, and the period map can be identified with the natural map  $Cr$ . In particular, the degree of  $Cr$  is equal to  $2^8 \cdot 51$ .*

**Corollary (8.7) (Coble [4]).** *The number of distinct projective equivalence classes of Cayley symmetroids which are regular-Cremona-equivalent to a fixed general Cayley symmetroid is equal to  $2^8 \cdot 51$ .*

The reader will notice that the key step in the proof of Theorem (8.5) was Lemma (8.3). This lemma was implicitly used by Coble [4]. Cossec and Dolgachev [6] have recently clarified the relationship between the Weyl group  $W_{2,4,6}$  and the Reye congruences, and they proved that the automorphism group of the generic "nodal" Enriques surface is isomorphic to the group  $\bar{W}(2)$  which was described in Lemma (8.3). Note that the covering K3 surface of such a "nodal" Enriques surface is the minimal desingularization  $S$  of a Cayley symmetroid  $X$ .

Coble [3] also proves that under regular Cremona transformations of  $P^3$ , a set of eight points which are the base points of a net of quadrics in  $P^3$  are transformed to 36 distinct projective equivalence classes of such sets of eight points. This fact is related to the period map  $p_{1,10}$  for K3 surfaces of Todorov type (1, 10), which has degree 36.

We close this paper by posing a problem. For  $K3$  surfaces of Todorov type  $(\alpha, k)$ , let  $N = k - 7$  so that the linear system  $|\mathcal{L}|$  maps  $X$  to  $\mathbf{P}^N$ . Does there exist any explanation of the degree of the period map  $p_{\alpha, k}$  in terms of regular Cremona transformations of  $\mathbf{P}^N$ , when  $N$  is greater than 3?

### References

- [1] E. Brieskorn, Die Milnorgitter der exzeptionellen unimodularen Singularitäten, *Bonner Math. Schriften*, **150** (1983).
- [2] F. Catanese and O. Debarre, Surfaces with  $K^2=2$ ,  $p_g=1$ ,  $q=0$ , preprint.
- [3] A. B. Coble, Point sets and allied Cremona groups (Part II), *Trans. Amer. Math. Soc.*, **17** (1916), 345–385.
- [4] —, The ten nodes of the rational sextic and of the Cayley symmetroid, *Amer. J. Math.*, **41** (1919), 243–265.
- [5] —, *Algebraic Geometry and Theta Functions*, Colloquium Publications, vol. X, Providence: American Mathematical Society (1929; 2nd edition, 1961).
- [6] F. Cossec and I. Dolgachev, On automorphisms of nodal Enriques surfaces, *Bull. Amer. Math. Soc.*, **12** (1985), 247–249.
- [7] I. V. Dolgachev, On Severi's conjecture concerning simply connected algebraic surfaces, *Dokl. Akad. Nauk SSSR*, **170:2** (1966), 249–252; *Soviet Math. Dokl.*, **7** (1966), 1169–1172.
- [8] —, Weyl groups and Cremona transformations, *Proc. Symp. Pure Math.*, **40:1** (1983), 283–294.
- [9] A. H. Durfee, Bilinear and quadratic forms on torsion modules, *Adv. in Math.*, **25** (1977), 133–164.
- [10] A. Kawauchi and S. Kojima, Algebraic classification of linking pairings on 3-manifolds, *Math. Ann.*, **253** (1980), 29–42.
- [11] V. S. Kulikov, Epimorphicity of the period mapping for surfaces of type  $K3$  (Russian), *Uspehi Mat. Nauk.*, **32:4** (1977), 257–258.
- [12] E. Looijenga and C. Peters, Torelli theorems for Kähler  $K3$  surfaces, *Compositio Math.*, **42** (1981), 145–186.
- [13] E. Looijenga and J. Wahl, Quadratic functions and smoothing surface singularities, *Topology*, **25** (1986), 261–291.
- [14] A. Mayer, Families of  $K3$  surfaces, *Nagoya Math. J.*, **48** (1972), 1–17.
- [15] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Berlin-Heidelberg-New York: Springer-Verlag (1973).
- [16] R. Miranda, The moduli of Weierstrass fibrations over  $\mathbf{P}^1$ , *Math. Ann.*, **255** (1981), 379–394.
- [17] —, Nondegenerate symmetric bilinear forms on finite abelian 2-groups, *Trans. Amer. Math. Soc.*, **284** (1984), 535–542.
- [18] R. Miranda and D. R. Morrison, The number of embeddings of integral quadratic forms, II, *Poc. Japan Acad.*, **62 Ser. A** (1986), 29–32.
- [19] D. R. Morrison, Some remarks on the moduli of  $K3$  surfaces, *Classification of Algebraic and Analytic Manifolds*, Progress in Math., **39**, Boston-Basel-Stuttgart: Birkhäuser (1983) 303–332.
- [20] —, On the moduli of Todorov surfaces, preprint.
- [21] V. V. Nikulin, Finite groups of automorphisms of Kählerian surfaces of type  $K3$ , *Trudy Mosk. Mat. Ob.*, **38** (1979), 75–137; *Trans. Moscow Math. Soc.*, **38** (1980), 71–135.
- [22] —, Integral symmetric bilinear forms and some of their applications, *Izv. Akad. Nauk. SSSR*, **43** (1979) 111–177; *Math. USSR-Izv.* **14** (1980), 103–167.

- [23] U. Persson, On degenerations of algebraic surfaces, *Mem. Amer. Math. Soc.*, **189** (1977).
- [24] I. Piateckii Shapiro and I. R. Shafarevich, A Torelli theorem for algebraic surfaces of type  $K3$ , *Izv. Akad. Nauk. SSSR*, **35** (1971) 530–572; *Math. USSR-Izv.* **5** (1971), 547–587.
- [25] M. Suzuki, *Group Theory I*, Berlin-Heidelberg-New York: Springer-Verlag (1982).
- [26] A. N. Todorov, A construction of surfaces with  $p_g=1$ ,  $q=0$  and  $2 \leq (K^2) \leq 8$ : counterexamples of the global Torelli theorem, *Invent. Math.*, **63** (1981), 287–304.
- [27] C. T. C. Wall, Quadratic forms on finite groups and related topics, *Topology*, **2** (1963), 281–298.

D. R. Morrison  
*Department of Mathematics*  
*Duke University*  
*Durham, NC 27706*  
*U.S.A.*

M.-H. Saito  
*Institute of Mathematics*  
*Faculty of Education*  
*Shiga University*  
*Otsu 520, Japan*