

**Kähler Metrics  
and  
Birational Geometry**

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**Hodge Theory in a New Century  
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# Hodge and Birational Geometry

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From Hodge's 1950 ICM address: "The results which I have described fall into two classes. In the one [. . .] In the other, the fact that the manifold carries a Kähler metric is fundamental [. . .] Yet many of the results do not depend on the actual Kähler metric selected [. . .]."

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The first part of the talk focusses on birational geometry during the past 50 years.

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- If  $X \subset \mathbb{C}^{n+1}$  is a hypersurface with a point of multiplicity  $k$  at the origin whose leading terms are a nondegenerate homogeneous polynomial of degree  $k$ , then the blowup replaces the origin by the corresponding hypersurface of degree  $k$  in  $\mathbb{P}^n$ .

## Examples, II: Cremona Transformations

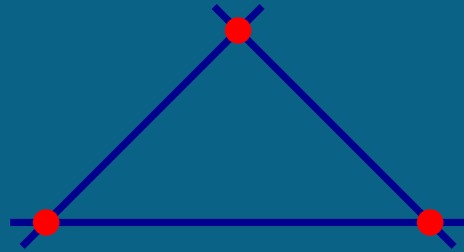
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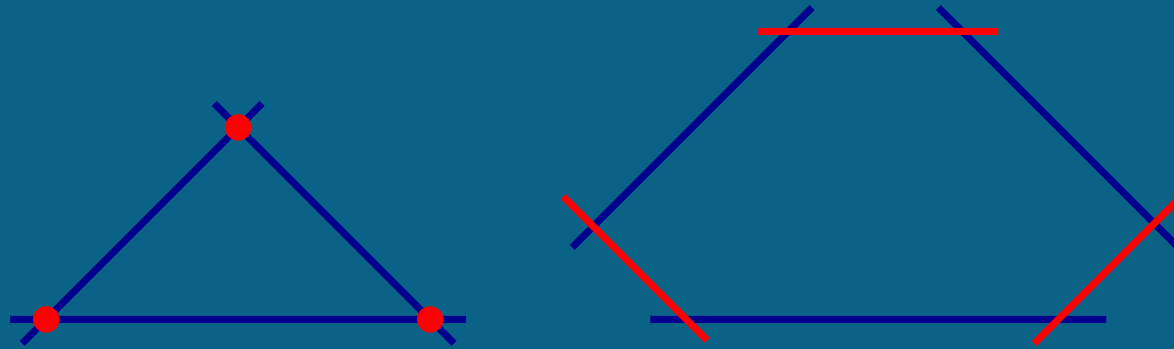
- Algebraic curves were studied by the classical Italian school of algebraic geometry as plane curves, and it was recognized that there were many different plane models of a given curve.
- These could be related by means of *Cremona transformations*, the simplest of which is described in projective coordinates by

$$[x, y, z] \mapsto \left[ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] = [yz, xz, xy].$$

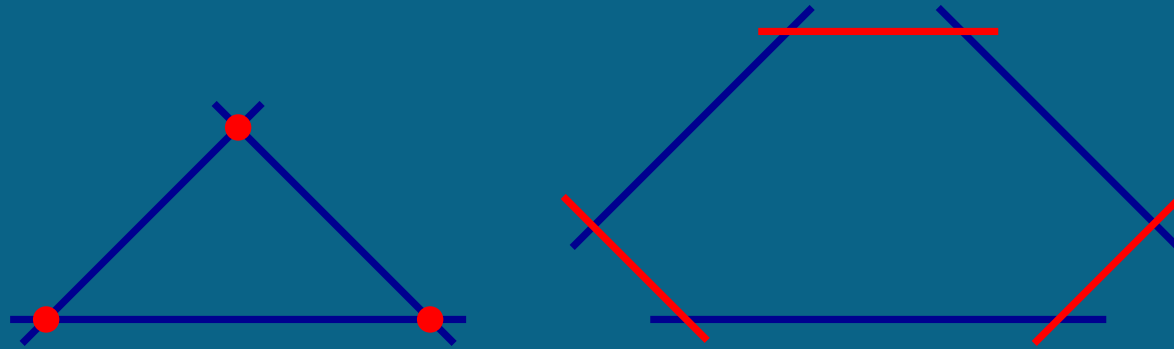
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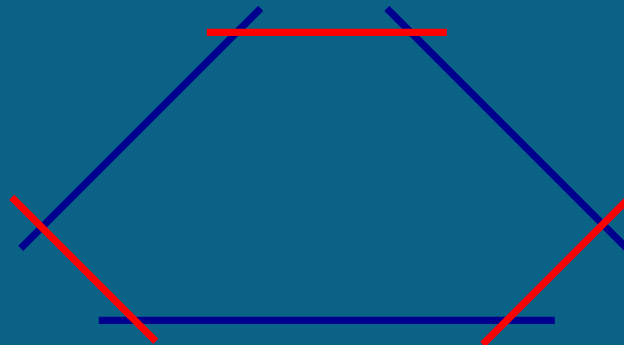
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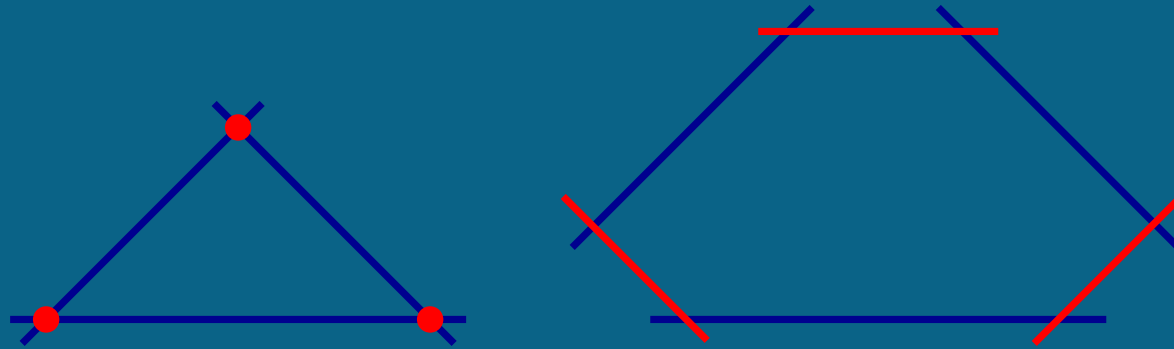
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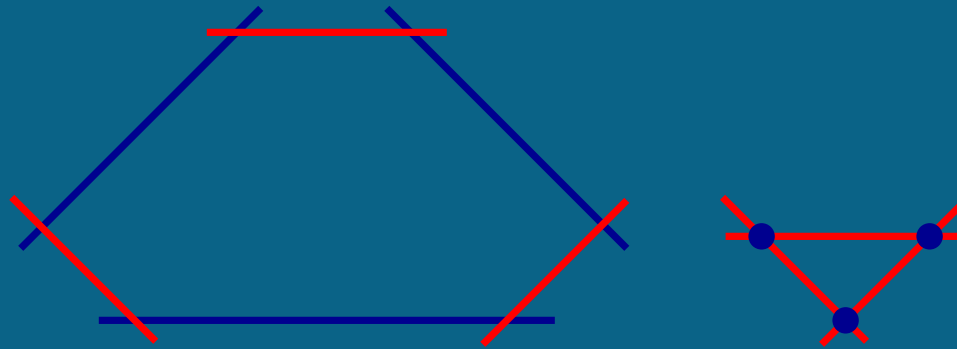
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## Examples, III: Flips and Flops

- $\mathbb{C}^2/\mathbb{Z}_2$  can be described as a hypersurface; the invariants  $s_1 = x_1^2$ ,  $s_2 = x_1x_2$ ,  $s_3 = x_2^2$  satisfy the equation  $s_1s_3 = s_2^2$ . And indeed the degree 2 hypersurface in  $\mathbb{P}^2$  is isomorphic to  $\mathbb{P}^1$ .

This singularity has a deformation  $s_1s_3 = s_2^2 + t$ , and Atiyah discovered in 1958 that upon making the basechange

$$t = -u^2,$$

the family of singularities can be *simultaneously resolved*.

To do this, let  $X_0$  be the basechanged family, factor the equation of  $X_0$  as

$$s_1 s_3 = (s_2 + u)(s_2 - u),$$

and blow up the ideal

$$s_1 = s_2 + u = 0$$

to produce  $X_-$ . This replaces the origin in  $X$  by  $\mathbb{P}^1 \subset X_-$ , giving a “small resolution” of the singularity.

Alternate description: blowing up the origin in  $X_0$  produces  $X$ , in which the origin has been replaced by a quadric hypersurface  $Q \subset \mathbb{P}^3$ , which is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . It turns out that the blowup of  $X_-$  along  $\mathbb{P}^1$  replaces  $\mathbb{P}^1$  by  $\mathbb{P}^1 \times \mathbb{P}^1$  which is isomorphic to  $Q$ , and so we get the usual blowup  $X$ . The fibers of the map  $X \rightarrow X_-$  give one of the two rulings of  $Q$ .

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The small resolution is ambiguous: if we blow up  $s_3 = s_2 + u = 0$  instead, we get  $X_+$ , again with a  $\mathbb{P}^1$  which can be blown up to yield  $X \rightarrow X_+$ ; the fibers of this map give the second ruling of  $Q$ .

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The birational map  $X_- \dashrightarrow X_+$  is the simplest example of a *flop*.



- To get the simplest example of a *flip*, start with  $X_0$  as above, defined by

$$s_1 s_3 = (s_2 + u)(s_2 - u),$$

and consider the  $\mathbb{Z}_2$ -action

$$(s_1, s_2, s_3, u) \mapsto (-s_1, -u, s_3, -s_2).$$

The equation of the hypersurface transforms by  $-1$ , and we let  $Y_0 = X_0/\mathbb{Z}_2$ .

The invariants of this  $\mathbb{Z}_2$ -action are generated by

$$v_1 = s_1^2, v_2 = s_1(s_2 + u), v_3 = (s_2 + u)^2, v_4 = s_3, v_5 = s_2 - u$$

subject to the relation  $v_1 v_3 = v_2^2$ . The equation of the hypersurface can be multiplied by either  $s_1$  or  $(s_2 + u)$  to yield two invariant equations

$$s_1^2 s_3 - s_1(s_2 + u)(s_2 - u) = v_1 v_4 - v_2 v_5,$$

$$(s_2 + u)s_1 s_3 - (s_2 + u)^2(s_2 - u) = v_2 v_4 - v_3 v_5.$$

We thus get a codimension two singularity in  $\mathbb{C}^5$  defined by three equations.

There are two small blowups  $Y_-$  and  $Y_+$ , constructed exactly as before.  $Y_-$  in fact has a singular point (of type  $\mathbb{C}^3/\mathbb{Z}_2$ ) whereas  $Y_+$  is nonsingular. The birational map  $Y_- \dashrightarrow Y_+$  is the simplest example of a flip.

## Non-separatedness in Moduli Spaces

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The existence of two different small resolutions of a double point in dimension three leads to non-separatedness in moduli spaces of surfaces.

Moduli of complex K3 surfaces: captured by period integrals (Torelli-type theorem), but not separated. At certain points in the moduli space, the Kähler cone suddenly shrinks to a sub-cone, and there is a choice of sub-cones.

# Resolution of Singularities

**Hironaka's Theorem:** Given a singular complex analytic space  $X_0$ , there is a sequence of blowups at smooth centers  $X_{i+1} \rightarrow X_i$  with  $X_N$  nonsingular.

## Minimal Models

The exceptional set  $C$  for the blowup of a point on a surface has a numerical characterization:  $K \cdot C = C \cdot C = -1$ . These are called “exceptional curves of the first kind.”

**Enriques' Theorem:** Given a nonsingular complex projective surface  $X_N$ , there is a sequence of blowdowns  $X_{i+1} \rightarrow X_i$  with  $X_i$  nonsingular, such that  $X_0$  has no exceptional curves of the first kind.

In higher dimensions, we are forced to allow singularities.

**Mori's Theorem:** Given a complex projective threefold  $X_N$  with at worst terminal singularities, there is a sequence of birational maps  $X_{i+1} \dashrightarrow X_i$  (projective blowdowns and flips), with each  $X_i$  having at worst terminal singularities, such that either  $K_{X_0} \cdot C \geq 0$  for all curves  $C$  on  $X_0$ , or  $X_0$  admits a nontrivial fibration  $X_0 \rightarrow B$  whose fibers contain all such curves  $C$ .

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Technical tool: The cone of effective curves and its dual, the cone of ample divisors. i.e., the cone of Hodge metrics. There will be sequences of Hodge metrics approaching the boundary of the cone, which in the limit realizes the contraction of the extremal curve.

## Factorization of Birational Maps

**Włodarczyk's Theorem:** Any birational equivalence  $X_N \dashrightarrow X_0$  between nonsingular complete varieties over  $\mathbb{C}$  can be factored into a sequence  $X_{i+1} \dashrightarrow X_i$ , where each map is either a blowup or a blowdown with a smooth center, and each  $X_i$  is nonsingular.

## Cobordisms in Morse Theory

Let  $W$  be a cobordism between two differentiable manifolds  $X_-$  and  $X_+$  and let  $f : W \rightarrow [a, b]$  be a Morse function with  $X_- = f^{-1}(a)$ ,  $X_+ = f^{-1}(b)$ .

As a variant of the usual setup, we glue  $X_- \times (-\infty, a]$  to the left side of  $W$  and  $X_+ \times [b, \infty)$  to the right side of  $W$  to obtain  $W'$  with a Morse function  $f' : W' \rightarrow (-\infty, \infty)$ .

Now  $\text{grad}(f')$  defines an action of  $\mathbb{R}_{>0}^*$  on  $W'$ , and we can recover  $X_-$  and  $X_+$  as follows: let

$$W'_- := \{x \in W' \mid \lim_{t \rightarrow 0} tx \text{ does not exist}\}$$

$$W'_+ := \{x \in W' \mid \lim_{t \rightarrow \infty} tx \text{ does not exist}\}$$

These are open subsets of  $W'$ , and  $X_{\pm} = W'_{\pm} / \mathbb{R}_{>0}^*$ .

Note that the critical points of the Morse function are the fixed points of the  $\mathbb{R}_{>0}^*$ -action.

## Birational Cobordisms

A *birational cobordism* between  $X_-$  and  $X_+$  is a variety  $B$  with an action of  $\mathbb{C}^*$  such that the sets

$$B_- := \{x \in B \mid \lim_{t \rightarrow 0} tx \text{ does not exist}\}$$

$$B_+ := \{x \in B \mid \lim_{t \rightarrow \infty} tx \text{ does not exist}\}$$

are nonempty and open, and  $X_{\pm} = B_{\pm}/\mathbb{C}^*$  (as geometric quotients).

## Examples

- $\mathbb{C}^*$  acting on  $\mathbb{C}^{n+1}$  as

$$(x_1, \dots, x_n, x_{n+1}) \mapsto (tx_1, \dots, tx_n, t^{-k}x_{n+1})$$

relates  $X_- = \mathbb{C}^n / \mathbb{Z}_k$  to  $X_+ = \mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ .

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relates  $X_- = \mathbb{C}^n / \mathbb{Z}_k$  to  $X_+ = \mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ .

(The limits on one side exist when  $x_1 = \dots = x_n = 0$ , and on the other side when  $x_{n+1} = 0$ .)

- $\mathbb{C}^*$  acting on  $\mathbb{C}^4$  as

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$$(x_1, x_2, x_3, x_4) \mapsto (t^2x_1, tx_2, t^{-1}x_3, t^{-1}x_4)$$

gives the simple flip.

## Examples, con.

In the case of a Cremona transformation, can follow the general procedure: let  $T_{\pm} \subset X_{\pm}$  be the coordinate triangles. There is a natural isomorphism (which acts as  $t \mapsto 1/t$  on the fibers) between

$$\pi_{-}^{*}(\mathcal{O}_{X_{-}}^{\times}(-T_{-})) \text{ and } \pi_{+}^{*}(\mathcal{O}_{X_{+}}^{\times}(T_{+})),$$

which we use to glue  $\mathcal{O}_{X_{-}}(-T_{-})$  to  $\mathcal{O}_{X_{+}}(T_{+})$ , yielding the space  $B$ .

## Symplectic reduction

The group  $\mathbb{C}^*$  contains a compact subgroup  $U(1)$ , and we can study the quotient by means of the moment map

$$\mu : B \rightarrow \mathfrak{u}(1)^* \cong \mathbb{R}.$$

For example, in the case of the simple flop

$$\mu(x_1, x_2, x_3, x_4) = \frac{1}{2}(|x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2),$$

and one finds  $X_a := \mu^{-1}(a)/U(1) \cong X_-$  for  $a \ll 0$  while  $X_a := \mu^{-1}(a)/U(1) \cong X_+$  for  $a \gg 0$ .

Given any Kähler metric on  $B$ , we can average over  $U(1)$  to obtain a  $U(1)$ -invariant metric. Then for each value of  $a$ , the space  $X_a$  inherits a Kähler metric.

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In the example, when  $a < 0$  the 3-sphere  $|x_3|^2 + |x_4|^2 = -a$  has volume proportional to  $(-a)^{3/2}$  and the  $U(1)$  fibers have length proportional to  $(-a)^{1/2}$ , so the area of  $\mathbb{P}^1 = S^3/U(1)$  is proportional to  $-a$ .

Similarly, when  $a > 0$  using the 3-sphere  $|x_1|^2 + |x_2|^2 = a$  leads to  $\mathbb{P}^1$  with area proportional to  $a$ .

## Adjacent cones

One of the lessons to be learned from this analysis is that metrics can be used to move from the Kähler cone into an adjacent cone containing the Kähler classes on a flipped or flopped space.

Kawamata: consider *moveable divisors* whose linear system  $|D|$  has no fixed components and  $n$ -dimensional image.

## Kähler metrics and birational geometry

- By including a Kähler class along with the complex structure, one gets a separated moduli space for K3 surfaces.
- A blowdown can be accompanied by a family of Kähler metrics in which the exceptional set shrinks to zero volume in the limit.
- A flip or flop can be accompanied by a family of Kähler metrics (from the birational cobordims) which leads from the Kähler cone to an adjacent cone.

## Complexifying the Kähler class

From string theory we learn that the Kähler class, while important in itself, should also be accompanied by a so-called  $B$ -field, taking values in  $H^2(X, U(1))$ . General parameter space looks like  $(\mathcal{K} + iH^2(X, \mathbb{R})) / H^2(X, \mathbb{Z})$ .

This is actually only an approximation to the “correct” parameter space for string theory which is valid when the volume of the Kähler metric is sufficiently large.

## Mirror symmetry

One of the big surprises from string theory is the appearance of Hodge's periods, in a completely unexpected context.

The period integrals  $\int_{\gamma} \Omega$  on a Calabi–Yau threefold, together with their derivatives with respect to the parameters, can be used to parameterize the moduli space. Mirror symmetry predicts that this same parametrization can be computed in terms of complexified Kähler parameters of a second, “mirror”, Calabi–Yau threefold.

As is by now well-known, an appropriate expression in Hodge's periods is given a power series expansion, and the coefficients are interpreted as (in physics) giving "quantum corrections to the classical cohomology algebra" or (in mathematics) counting numbers of rational curves on the Calabi–Yau threefold in a fixed cohomology class.