A deformation invariant of 2D SQFTs

based on 1904.05788 w/ D. Gaiotto
and 1902.10249 w/ DG. + E. Witten
(and also 1811.00589 w/ DG)

General question: What is the homotopy type of "the" space of (your favorite type of) qft?

My favorite: 2D, minimally supersymmetric $\mathcal{N}=(0,1)$
(in (1+1)D, or 2d)

unitary, compact qfts.

Idea of compactness:
a spectral condition

examples: Sigma models with compact target.

Quantum mechanics = 1D QFT
Sigma model: quantum particle in a ~~space~~ manifold.

2

$\mathcal{H} = L^2(M)$

$\hat{H} = \Delta$

Compact $\longleftrightarrow$ discrete spectrum

Or a sigma model on a noncompact target
with potential energy not too slow at ∞

(So the harmonic oscillator is compact)

Rational CFTs should be compact.

In the QM case, can define compact to mean that the
Wick rotated partition function converges: $\exp(-\tau \hat{H})$ trace
class for $\tau > 0$

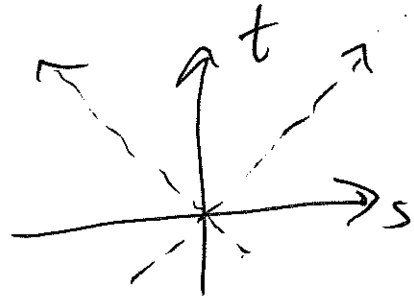
$\mathcal{N} = (0,1)$ means:

any 2D QFT has, among its generators

$\hat{H}$ = hamiltonian (generates flow in time direction $\frac{\partial}{\partial t}$)

$\hat{P}$ = momentum (- - - - - space - - $\frac{\partial}{\partial \xi}$)

In Lorentz signature: light cone $t+s, t-s$



$N=(0,1) \Rightarrow \exists$ a fermionic operator $\bar{G}$ s.t.

$$\bar{G}^2 = H + P \quad \left(\text{generator of } \frac{\partial}{\partial \bar{z}}\right)$$

All ops self-adjoint ... need $(\sqrt{-1})$ power $\bar{G}^2$ ---

(suppress this)

$\bar{G}^2$ mean $[\bar{G}, \bar{G}]$

SQFTS = space of such theories

A standard way physicists protect qfts from being connected by a path is through "anomalies"

In 2D, $\exists$ a "gravitational anomaly" $c_R - c_L$

! c_R, c_L are not deformation invariants

but $c_R - c_L \in \mathbb{R}$ is

For bosonic theory $c_R - c_L \in 8\mathbb{Z}$

For theories with fermions, $c_R - c_L \in \frac{1}{2}\mathbb{Z}$.

Use $n = 2(c_R - c_L) \in \mathbb{Z}$. (which will end up being a homotopical degree)

So $SQFT = \coprod_{n \in \mathbb{Z}} SQFT_n$

<u>Motivation for this talk</u>

$SQFT$ is a $\mathbb{Z}$ graded ring (because we can add + multiply field theories)

$SQFT_n$ is naturally an E_∞ ring Ω-spectrum.

Hypothesis: $SQFT_\bullet = TMF_\bullet$

leads to lots of predictions

$SQFT_n$ has a "renormalization group" flow

far UV

far IR

C-theorem basically says that this is a Morse-Bott flow.

critical pts = SCFTs

probably a fin-dim'l manifold.

(whereas $SQFT_n$ is ∞-dim'l)

Working physicists' def for the topology on $SQFT_n$:

two SQFT's are <u>deformation equivalent</u> if they are related by

- usual homotopy deformations (marginal, irrelevant operators)
- RG flow
- IR equivalent (with IR limit being compact)

(6)

In the presence of SUSY, interesting physics questions include

$\mathcal{F} \in \mathrm{SQFTn}$

(i) does SUSY spontaneously break in $\mathcal{F}$?
(i.e. is the IR theory the zero theory)

(ii) does $\mathcal{F}$ have a small deformation which spontaneously breaks SUSY?

(iii) Is $\mathcal{F}$ deformation equivalent to a theory with spontaneous symmetry breaking?

e.g. Field Content

scalar multiplet:
- R-valued noncompact boson φ
- its (right-moving) superpartner $\overline{\psi}$

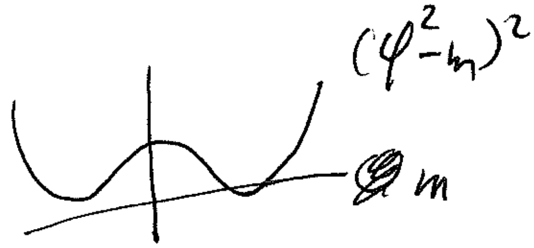
- left-moving fermion λ
(superpartner is auxiliary)

Superpotential $W = \lambda(\varphi^2 - m)$

$$\text{Lagrangian} = \varphi \Delta \varphi + \overline{\psi} \partial \overline{\psi} + \lambda \overline{\partial} \lambda + (\varphi^2 - m)^2 + \lambda \varphi \overline{\psi}$$

If $m<0$, we are in situation (i).

If $m>0$,



the far IR get trivial $\oplus$ arf

(equiv. to one with SUSY breaking)

$\Rightarrow$ goes to (iii)

If $m=0$, flows to a CFT, # (ii)

sigma model with target round S^3

$\mathcal{N}=(0,1)$

unmatched fermions $\rightsquigarrow$ possible anomaly

To fully define the sigma model, need

anomaly cancellation data

$\underset{=}{\text{mathematics}}$ "string structure"

$\underset{=}{\text{physics}}$ "quantum field"!

If it exists, gives a torsor for H^3(target, $\mathbb{Z}$).

In the S^3 case, if it exists it is canonically trivialis

$S^3_k = S^3$ sigma model with string structure $= k$

What is the for IR?

Methods: · Write down every symmetry you can think of
· write down every CFT you can find of with those symmetries.

+ anomaly-match

For the problem the for IR limit of S^3_k, get

$\mathcal{N}=(0,1)$ WZW model for $SU(2)$

if $k \neq 0$, then have left currents $J_{1,2,3}$

right currents $\bar{J}_{1,2,3}$

fermi $\bar{\psi}_{1,2,3}$

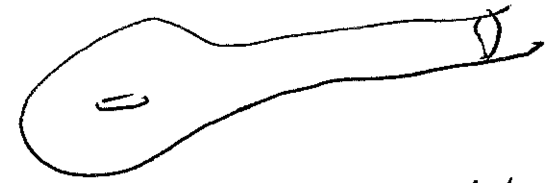
{left} bosonic WZW level $|k|-1$

{right, fermions} $SU(2)_{|k|+1}$

· SUSY spontaneously broken if $k=0$ (Case (i))

K3 surface has no string structure.

(K3 surface $\setminus \{pt\}$)



not a compact thing, but a valid sigma $\mathcal{X}$

Can project to $\mathbb{R}$, corresponds to operator Φ in $\mathcal{X}$.

Take slice of Φ ("fibers") using Lagrange multiplier

$\mathcal{X}$ + (fermi multiplet λ)

with new superpotential term

$$\lambda(\Phi - m)$$

If $m = +\infty$, get fiber $= S^3_{24}$ sigma model

If $m = -\infty$, get nothing.

$\Rightarrow S^3_{24}$ has property (iii)

How to protect $\mathcal{F} \in$ SQFT from being null homotopic?
Need a deformation invariant of SQFTs which is

- Computable
- vanishing if susy not spontaneously broken
- $\neq 0$ for $\mathcal{F}$

Famous example: elliptic/Witten genus

$= Z_{KR}(\mathcal{F})$, the partition function of $\mathcal{F}$ on flat tori with non-bounding spin structure (A 3 real dim'l space.) with natural coords $\tau, \bar\tau$, volume.

Compactness $\Rightarrow Z_{KR}$ converges [with normalization factor of $\eta(\tau)^*$]

Let us assume real-analyticity as a property of our theories.

Standard facts

(i) $\dfrac{\partial Z_{KR}}{\partial \bar\tau} = \dfrac{\partial Z_{KR}}{\partial \text{vol}} = 0$

$\rightarrow$ ~~[illegible]~~ weakly holomorphic modular form of wt. $-\frac{d}{2}$

weakly: pole at cusp

(ii) q-expansion $\in \mathbb{Z}[[q]]$

one-point function of stress-energy tensor

Why? (i) $\dfrac{\partial Z_{KR}}{\partial \bar\tau} \propto \langle T_{\bar z \bar z} \rangle_{KR}$

$= \langle \bar G[\bar G_{\bar z}] \rangle_{KR}$

$= \langle \bar G[0] \rangle_{KR} = 0$

For ref, use $T_{\bar z \bar z}$, etc.

(ii) I can compute Z_{RR}(torus) by first computing on S^1_R

→ S^1-equivariant SQM model and Z_{RR} = index $\in \mathbb{Z}$

Note: (i) + (ii) ⇒ Z_{RR} is a deformation invariant.

What if $Z_{RR} = 0$?

Problem: ∄ weakly holo modular form of wt $3/2$. But S^3_k

has $c_R - c_L = \frac{3}{2}$. ⇒ $Z_{RR} \equiv 0$.

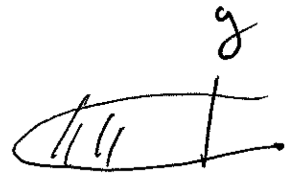
$y \in SQFT_n$, $Z_{RR}(y) = 0$

"

You in a family with spontaneous susy breaking

dynamicalize the parameter in the theory.

get a non-compact thing.



Try $Z_{RR}(\mathcal{X})$... probably converges – contributions from end is $0 \cdot \infty \to 0$.

(1) Is it holomorphic? (i.e. depend on $\bar\tau$?)

$$\frac{\partial Z_{RR}(\mathcal{X})}{\partial \bar\tau} \propto \langle T_{\bar z \bar z} \rangle = \langle \bar G [\bar G_{\bar z}] \rangle = \text{boundary term}$$

[illegible] through example:

$$\sqrt{\operatorname{Im}\tau}\, \frac{\partial Z_{RR}(\mathcal{X})}{\partial \bar\tau} = (\text{factors of } \sqrt{-1} \text{ and } \eta(\tau)) \cdot \langle \bar G_{\bar z} \rangle_y$$

and $\frac{\partial Z_{KR}}{\partial \mathrm{vol}} \propto \langle \bar{G}_2 \rangle$, $= 0$ if $\mathcal{Y}$ is SCFT.

(ii) Let $\hat{f}(\tau,\bar{\tau}) \sim Z_{KR}(x)$

Let $f(\tau) = \lim_{\bar{\tau} \to i\infty} \hat{f}(\tau,\bar{\tau})$ (this happens when anomaly on S^1_R)

Do q-expansion of f.

$\in \mathbb{Z}((q))$ up to a correction related to Atiyah-Patodi-Singer, which often vanishes.

Summary

Take $\mathcal{Y} \in SQFT_n$ with $Z_{KR}(\mathcal{Y}) = 0$. Assume $\mathcal{Y} \in SCFT$.

Solve for $\hat{f}(\tau,\bar{\tau})$ with $\sqrt{8c_2}\,\frac{\partial \hat{f}(\tau,\bar{\tau})}{\partial \bar{\tau}} = (\text{factors}) \langle \bar{G}_2 \rangle_{\mathcal{Y}}$

real antiholMF class $[\hat{f}]$ is uniquely determined by $\mathcal{Y}$.

q-expand: $[f] \in \mathbb{C}((q))/MF$

If $\mathcal{Y}$ is null-homotopic in not 0, but it is integral..

take further quotient $\mathbb{C}((q))/MF + \mathbb{Z}((q))$

For WZW model, level $k-1$, $k+1$ get $f = -kG_2 + \mathbb{Z}[[q]]$

or Eisenstein series

$\Rightarrow S^3_k$ is null-homotopy iff $k \in 24\mathbb{Z}$