Limits of K3 metrics

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Gromov–Hausdorff limits

- (Pointed) Gromov–Hausdorff limits exist under very general circumstaces, such as lower bounds on Ricci curvature.
- The idea is to take isometric embeddings of two spaces and measure the distance between specified points.
- Minimizing over all choices of embeddings gives the distance.
- The set of all pointed metric spaces is quite well behaved under such limits.
- The corresponding theory in physics is M-theory compactified on our manifold, and we will sometimes invoke a physical duality (which always has a geometric counterpart) to explain what is going on.

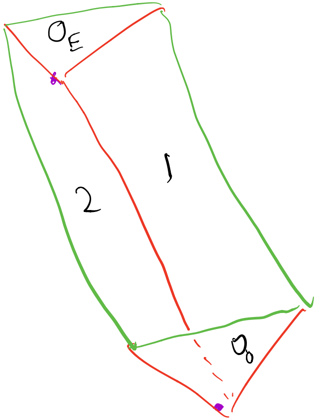
K3 metrics

- To each volume one, Ricci-flat metric on the K3 manifold X, we can associate the self-dual 3-plane in $H^2(X, \mathbb{R})$, which is a space of signature $(3, 19)$. This give a point in the locally symmetric space $\mathcal{M}_1 := \Gamma \backslash O(3, 19)/O(3) \times O(19)$, where Γ is the component group of $\mathrm{Diff}(X)$.
- It is known that the point in $\mathcal{M}_1$ determines the volume one metric up to diffeomorphism, and that the image is an open subset of $\mathcal{M}_1$ (the complement of a set of codimension 3).
- Moreover, the "missing" points correspond to singular K3 spaces, with orbifold-type singularities.
- Gromov–Hausdorff theory suggests a space $\mathcal{X}_1 \to \mathcal{M}_1$ labeling pointed metrics of volume 1, and this space exists.
- We can extend the moduli space to include all metrics, and extend the universal fibration to $\mathcal{X} \to \mathcal{M}$.

Compactifying the moduli space

- As a first attempt at compactifying the moduli space, we will use the Borel–Serre compactification of $\mathcal{M}_1$. It does not give the right answer, but it points in the right direction.
- The Borel–Serre compactification has boundary components of codimension one associated to parabolic subgroups.
- More concretely, the codimension one boundary components are associated with primitive, totally isotropic sublattices $\Lambda \subset \mathbb{Z}^{3,19}$. Given the totally isotropic sublattice, there is a splitting $\Lambda^\perp = \Lambda \oplus H$ where H is an even unimodular lattice of rank $(k, 16+k)$, where $3-k$ is the rank of Λ.
- There are two possibilities when $k=0$ (the $E_8 \oplus E_8$ and $Spin(32)$ cases), and only one possibility for each other value of k. These are illustrated below.

The figure



Cheeger–Gromov & Cheeger–Tian

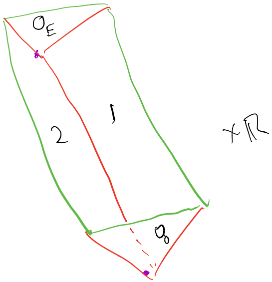
- In Gromov–Hausdorff theory, given a sequence of Ricci-flat metrics ds_j^2 (together with points P_j) which diverges with respect to the metric on $\mathcal{M}$, there is a subsequence which converges in the Gromov–Hausdorff distance. Far enough out on the subsequence, there is a fibration structure $\pi : X \to B$ away from finitely many points, where the fibers of π are nilmanifolds with specified volume growth.
- There are four cases of this, corresponding to the four cases of Borel–Serre boundaries. Before we get to that, we need to discuss the behavior of the volume.

The behavior of the volume

We can do three natural things, given a degenerating sequence:

- We can normalize the subsequence $ds^2_{j_k}$ so that $\mathrm{vol}(ds^2_{j_k}) \to V$ with $V \in (0, \infty)$. If the primary rate of growth dictates a fibration structure $\pi : X \to B$ with $\dim B = d$, let $v_k := \mathrm{vol}(ds^2_{j_k}|_{\pi^{-1}(\pi(P_{j_k}))})$ with $v_k \to 0$. Then the image B has volume approximately $V/v_k \to \infty$.
- Alternatively, we can send the overall volume to zero while retaining the structure of B in the limit. This is achieved by rescaling the metric in the previous case by v_k, i.e., $\widetilde{ds^2}_{j_k} = v_k\, ds^2_{j_k}$. Then B will approach something of volume V in the limit.
- On the other hand, we can send the overall volume to infinity while holding the general fibers of π at finite size. If we let $\widehat{ds^2}_{j_k} = ds^2_{j_k}/v_k$ then we see that the volume of the fiber $\pi^{-1}(\pi(P_{j_k}))$ approaches V.

Another figure



$r = 0$
codim 2 boundary
finite size B; don't need P_j

$r = V$
codim 1 boundary

$r = \infty$
codim 2 boundary
finite size fibre
(can scale either torus in isolation)

Case 2: elliptic fibrations (Gross–Wilson)

- The first case to consider is the case of elliptic fibrations, i.e., the case in which the fiber of π is a two-torus. If we normalize so that $r \to 0$, then we get Gross and Wilson's semi-flat metric in the limit. The base B is a two-sphere, and there are up to 24 points p_j. The metric has a special form (originally derived in Greene–Shapere–Vafa–Yau). The size of the two-torus fiber is also going to zero. The singular fibers are ALG spaces, maybe also ALH spaces.
- The physical dual is based on the fact that M-theory compactified on T^2, when the volume of T^2 is small, is dual to type IIB string theory compactified on S^1 with S^1 large. In the small T^2 limit, the space decompactifies along S^1. (This is the F-theory dual.)

Case 2: elliptic fibrations, con.

- In the finite volume limit, the T^2 fiber has volume $v_k \to 0$ while the twp-dimensional base has volume $V/v_k \to \infty$. There are three kinds of elements of T^2: there is a single two-cycle with both legs in the fiber, in which case the volume is $v_k \to 0$; there are four two-cycles with one leg in the base and one leg in the fiber, which have volume approximately $\sqrt{v_k}\sqrt{V/v_k} \equiv \sqrt{V}$; and there is a single two-cycle with both legs in the base, which has volume $V/v_k \to \infty$. When combined with the invariant two-cycles in the anti-self-dual directions, this is case $k = 2$ in the Cheeger–Gromov type classification.
- The singular fibers were classified by Kodaira: type I_n, I_n^*, II, III, IV, IV^*, III^*, II^*. Each has a specified metric behavior. Moreover, the class of the elliptic curve is algebraic for a compatible complex structure.

Case 1: fibrations by nilmanifolds (HSVZ)

- The metrics here were first constructed by Hein–Song–Viaclovsky–R. Zhang (preprint not yet ready). I heard about it in lectures by Viaclovsky and Zhang at the Simons Collaboration meeting a few weeks ago, and hopefully we will hear more when Viaclovsky visits in two weeks.
- There are a variety of examples, fibered over a dimension one base (which we can take to be $[0, 1]$ in the case of $r = 0$). The fibers are nilmanifolds; there are points p_j, $j = 1, \dots, n$, at which the nilmanifold type changes; there is a function $H : [0, 1] \to \mathbb{R}$ with $H(0) < 0$, $H(1) > 0$, $H(p_i - \varepsilon) + 1 = H(p_i + \varepsilon)$, subject to the constraint that $n \leq 18$. What's happening is that there is a certain amount of twist on the nilmanifold which is changing as you move from point to point.

Case 1: fibrations by nilmanifolds, con.

- I should have mentioned in the Gross–Wilson case that there are certain types of four-manifolds (gravitational instantons) in the neighborhoods of the special points. Similarly here, there are certain types of four-manifolds (gravitational instantons). At the endpoints of B, they are Tian–Yau manifolds, constructed by taking a (generalized) del Pezzo surface dP_k – a blowup of $\mathbb{P}^2$ in $k \leq 9$ points (or $\mathbb{P}^1 \times \mathbb{P}^1$ if $k = 1$) – a putting a complete Ricci-flat metric on the complement of a chosen anti-canonical curve $C \in |-K_{dP_k}|$. (The two ends are related by orientation reversal.)
- At the p_i's, one uses the Taub–NUT space to interpolate between the two different kinds of nilmanifold. At the transition, the nilmanifold is becoming long in the T^2 direction and short in the S^1 direction, whereas it is more uniform out in the middle.

Case 1: fibrations by nilmanifolds, p. 3

- HSVZ recently constructed degenerations of this type. They had the Taub–NUT spaces and the Tian–Yau manifolds, and they carefully showed that you can glue these together and then smooth out the resulting space to get a Ricci-flat K3 metric. There is a parameter involved in the smoothing which specifies the sizes of the circles, and those sizes tell you that the construction is OK as long as the circles are relatively small compared with the size of B. Thus, the construction works in a neighborhood of the boundary locus.
- To compute the effect on $H^2(X)$, we scale so that the two torus is of finite size while the circle shrinks to zero (letting the overall volume stay finite). So S^1 has volume $v_k \to 0$, T^2 has volume V, and B has volume $V/v_k \to \infty$.

Case 1: fibrations by nilmanifolds, p. 4

- Here are the two-cycles: we have two of volume $v_k\sqrt{V} \to 0$, having one leg in S^1 and one leg in T^2; we have two of volume V, one having both legs in T^2 while the other has a leg in S^1 and a leg in B^1, and two of volume $\sqrt{V}/v_k \to \infty$, having a leg in T^2 and a leg in B^1. This is why we get $k = 1$ for these cases.
- The physical duality in this case is to type I' (a variant to IIA). The shrinking S^1 allows us to move from M-theory to the type IIA string, and the dual description (T-dualizing along both S^1's) is in terms of the type I' string compactified on an inteval $[1, 0]$.

Gravitational instantons

- Gravitational instantons come in several varieties: ALE, ALF, ALG, ALH (and some things which do not quite satisfy the growth conditions).
- ALE spaces take the form $\mathbb{C}^4/\Gamma$ for subgroups $\Gamma \subset SL(2, \mathbb{C})$.
- ALF, ALG, ALH are similar, except that some of the directions at infinity are toroidal. One torus direction means an ALF space, and so on.

Case 0_O: fibrations by circles (Foscolo)

- This case corresponds to gravitational instantons of type ALF, which were classified by Minerbe and G. Chen–X. Chen. There are A and D types, but no E type. (And a few extra D types...)
- $\pi : X \to B$ collapses S^1.
- In $H^2(X)$, three cycles grow and three cycles shrink; nothing is left invariant. So $k = 0$. But which $k = 0$ case is it?
- The fact that there are no ALF spaces of type E stongly suggests that you can only realize the $Spin(32)/\mathbb{Z}_2$ lattice here, not the $E_8 \times E_8$ lattice.

Case 0_E: fibrations by three-tori (G. Chen–X. Chen)

- This case corresponds to gravitational instantons of type ALH, classified by Chen–Chen based on earlier work of Hein.
- $pi : X \to B$ has general fiber T^3 and base of dimension one.
- In $H^2(X)$, three cycles grow, three cycles shrink, and nothing is left invariant.
- Over B, there is an ALH space on each end, and those ALH spaces can include E type singularities. So it's natural to identify this case with 0_E.
- Two cases which are very different geometrically correspond to the two $k = 0$ components of the boundary.

Conclusions

- What I haven't had a chance to talk about today is the physical dualities for the cases $k \neq 2$. In the other cases, there are natural duals involving type I', the Horara–Witten dual of the heterotic string, and various other constructions. Maybe I'll come back to that in a future lecture.
- The duality which I haven't discussed in any of the cases is the duality to the heterotic string: taking the small volume limit of the K3 compactification of M-theory and turning it into the large volume limit of the heterotic string compactified on T^3.
- The latter can be analyzed perturbatively. The identification with the E_8 string relies in part on the fact that the Foscolo construction involves ALF instantons which cannot realize the E_8 gauge symmetry, but there is more to it than that. As T^3's shrink to zero size from the K3 side, there is an emergent E_8 gauge field which makes the duality even more explicit.