D.R. Morrison Supergeometry Lectures Witten 2018, UCSB, part II.

Based on:

Quantum Fields and Strings, A guide for mathematicians

Vol 2, Witten's Lectures 16, 17, 18

Supersymmetry algebras:

4D, 8 supercharges

→ "$\mathcal{N} = 2$ in 4D"

→ written in terms of

$\mathcal{N} = 1$ ~~superfields~~

Classify reps of super algebras

irrep = "multiplet"

decompose into Poincaré reps

"particle types"

Superspace: Find a supermanifold which extends $M^{1,3}$ or $\mathbb{R}^4$ to a space with even and odd coordinates.

Good ones for 2, or 4 supercharges

There is a "harmonic superspace" for 8-supercharges.

None known for > 8 supercharges

Gauge theories: $\mathcal{N} = 0$

$\mathcal{N} = 1$

→ $\mathcal{N} = 2$

4D Gauge theory:

$$\mathcal{L} = \frac{1}{4g^2} \int \mathrm{Tr}\, \overbrace{F \wedge * F}^{\|F\|^2}$$

Hodge Star

$A$ = $\mathfrak{g}$-valued gauge field.

$F = dA + A \wedge A$

= $\mathfrak{g}$ valued curvature (2-form)

These are invariant under section of $G$

All physical quantities are $G$-invariant.

$$\tilde{\mathcal{L}} = \frac{1}{4g^2} \int \mathrm{Tr}\, F \wedge * F + \frac{i\theta}{16\pi^2} \int \mathrm{Tr}\, F \wedge F$$

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$\mathcal{N} = 1$: • Curvature is part of a multiplet.

• gauge field is part of a multiplet

$(A, \lambda, D)$

$D$: bosonic, quadratic in action. "auxiliary field"

$\lambda$: spinor valued (fermion field)

$$\mathcal{L}_{\text{fermion}} = \frac{1}{4g^2} \int (\mathrm{Tr}\, F \wedge * F) + \frac{i\theta}{16\pi^2} \int \mathrm{Tr}\, F \wedge F + \sum_i \left( \int \bar{\varphi}_i \not{D} \varphi_i^* + \int \bar{\varphi}_i^* \not{D} \varphi_i \right),$$

$\varphi_i$ – left $\varphi_i^*$ – right

$n_f = n_f^*$

$\underline{\mathcal{N}=1}$:

$$\mathcal{L}_{\mathcal{N}=1} = \frac{1}{4g^2}\int \mathrm{Tr}\, F\wedge * F \quad \left(+ \text{ quadratic in } D\right)$$

$$+ \frac{i\theta}{16\pi}\int \mathrm{Tr}\, F\wedge F + \int \mathrm{Tr}\left(\bar{\lambda}_i \not{D} \lambda_i\right)$$

$\lambda$ = adjoint valued spinor

$\mathcal{N}=1$ has "chiral superfields"

$\Phi \quad (\varphi, \psi, \hat{F})$

Something like "$\varphi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}\hat{F}$"

complex scalar — fermion — quadratic

Term in Lagrangian:

$$\int d^4x\, d^4\theta\, \mathrm{Tr}\, \bar{\Phi}\Phi$$

$$+ \varepsilon\int d^4x\, d^2\theta\, \mathrm{Tr}\, \Phi\Phi$$

+ complex conjugate

+ fermion term

more generally, "superpotential polynomial" in $\Phi$.

$\underline{\text{Remark}}$: In $\mathcal{N}=2$, $\varepsilon=0$.

$\underline{\mathcal{N}=2 \ 4D}$

Curvature! $W$

- $\mathcal{N}=2$ vector multiplet $= \begin{pmatrix} (\mathcal{N}=1)\text{-vector}\ + \\ + (\mathcal{N}=1)\text{-chiral} \end{pmatrix}$

- $\mathcal{N}=2$ hypermultiplet

$\Phi$: $(\mathcal{N}=1)$-chiral + $(\mathcal{N}=1)$ chiral

$\underline{\text{curvature}}$ is in a spinor valued multiplet. Key component

$W_\alpha$ ($\mathfrak{g}$-valued)

$$\mathcal{L}_{\mathcal{N}=2} = \frac{i}{4\pi}\int d^4x\, d^4\theta\, \tau(\Phi) W_\alpha W^\alpha$$

$$+ \frac{1}{4\pi}\int d^4x\, d^4\theta\, K(\Phi, \bar{\Phi})$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

varies holomorphically with $\Phi$.

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$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{4\pi}\int_{M^4} d^4x \Big( i\big[\bar{\tau}(F^+\wedge F^+) - \tau(F^-\wedge F^-)\big]$$
$$+ K_{\varphi\bar{\varphi}}\, \partial\phi\, \partial\bar{\phi}$$
$$+ (\mathrm{Im}\,\tau)\, \bar{\lambda}\, i \not{D}_A \lambda$$
$$+ K_{\phi\bar{\phi}}\, \bar{\psi}\, i \not{D}_A \psi \Big)$$

$X \circlearrowleft G$

$X /\!/ G$

$K(\phi,\bar{\phi})$ = Kähler potential

$K_{\phi\bar{\phi}}\, d\phi\, d\bar{\phi}$ is a Kähler metric

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Sigma-Model with target $X$ = Kähler mfld

$\Phi : \mathbb{R}^4 \longrightarrow X$

$G \circlearrowleft X$, Physical Theory $X/G_{\mathbb{C}}$ (solution of D-terms$/G$)