Source: Quantum Fields & Strings: A Course For Mathematicians (vol. 2) Witten's lectures 16, 17, 18

Supersymmetry algebras:

Today is on 4D, 8 supercharges "$\mathcal{N}=2$ is 4D"

These are written in terms of $\mathcal{N}=1$ superfields.

Classify reps of SUSY algebras

- Irreps correspond to "multiplets"
- Decompose into Poincaré reps ~ "particle types"

Superspace: Find a supermanifold which extends $M^{1,3}$ or $\mathbb{R}^4$ to a space w/ even/odd coordinates.

These are invariant under action of G (Lie group ass. to $\mathfrak{g}$)

- All physical quantities are G-invariant

Now set:

$$\mathcal{L} = \frac{1}{4g^2}\int \mathrm{Tr}\, F\wedge * F + \frac{i\theta}{16\pi^2}\int \mathrm{Tr}\, F\wedge F$$

$\boxed{\mathcal{N}=1.}$

curvature
Gauge field } part of the multiplet

Fermion field (Spinor valued)

$$(A, \lambda, D)$$

Bosonic, quadratic in the action
"Auxiliary field"

$$\mathcal{L}_{\text{fermion}} = \frac{1}{4g^2}\int \mathrm{Tr}\, F\wedge * F + \frac{i\theta}{16\pi^2}\int \mathrm{Tr}\, F\wedge F$$

$$\sum\left(\int (\bar{\Psi}_i, \not{D}\psi_i^*) + \int \bar{\Psi}_i \not{D} \psi_i\right)$$

$(n_f = n_f^+)$

Ψ_i left ψ_i^+ right

$\mathcal{N}=1$

$$\mathcal{L}_{N=1} = \frac{1}{4g^2}\int \mathrm{Tr}\, F \wedge * F + \frac{i\theta}{16\pi^2}\int \mathrm{Tr}\, F\wedge F + \int \mathrm{Tr}(\bar\lambda i \not{D} \lambda)$$

+ something quadr. in D

(λ adjoint-valued spinor)

$\mathcal{N}=1$ also has:

Chiral superfields.

$$(\underbrace{\Phi}_{\text{complex scalar}}, \underbrace{\psi}_{\text{fermion}}, \underbrace{F}_{\text{auxiliary field}})$$

Term in Lagrangian:

$$\underbrace{\int d^4x\, d^4\theta\, \mathrm{Tr}\, \bar\Phi \Phi}_{\text{Kinetic}} + \epsilon\left(\int d^4x\, d^2\theta\, \mathrm{Tr}\, \Phi\Phi + \text{complex conjugate}\right)$$

+ Fermion term

More generally, "superpotential" polynomial in Φ

$\mathcal{N}=2$

$\mathcal{N}=2$ vector multiplet $\longleftrightarrow$ $\mathcal{N}=1$ vector multiplet & a
$\mathcal{N}=1$ chiral

$\mathcal{N}=2$ ~~vector~~ hypermultiplet $\longleftrightarrow$ $\mathcal{N}=1$ chiral + $\mathcal{N}=1$ chiral

Curvature is in a spinor-valued multiplet

Key component W_α ($\mathfrak{g}$-valued)

$$\mathcal{L}_{N=2} = \frac{i}{4\pi}\int d^4x\, d^4\theta\, \tau(\Phi)\, W_a W^a$$
$$+ \frac{1}{4\pi}\int d^4x\, d^4\theta\, K(\Phi, \bar{\Phi})$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$ ~ varies holomorphically w/ Φ

$$\mathcal{L}_{N=2} = \frac{1}{4\pi}\int_{M^4} d^4x \Big(i\,[\bar{\tau}(F^+ \wedge F^+) - \tau(F^- \wedge F^-)] + K_{\phi\bar{\phi}}\, \partial\phi\, \overline{\partial\phi} + (\operatorname{Im} \tau)\, \bar{\lambda}_i \gamma_A \lambda + K_{\phi\bar{\phi}}\, \bar{\psi}_i \gamma_A \psi \Big)$$

$$*F^{\pm} = \pm F^{\pm}$$

$K(\phi, \bar{\phi})$ = kähler potential.

$K_{\phi\bar{\phi}}\, d\phi\, d\bar{\phi}$ is a Kähler metric

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Sigma models; w/ target X a kähler manif.

$\Phi : \mathbb{R}^4 \longrightarrow X$

G acts on X

Physical theory only sees G-invariant things.

$X/G \supseteq$ Solution of D-terms $/ G$

$X \circlearrowleft G$

$X /\!\!/ G$ symplectic red.