The Abel-Jacobi map & Higher Chow Groups

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Let C be a proj. curve $/\mathbb{C}$. Period integrals give a map $C \longrightarrow \mathrm{Jac}(C) \cong \mathrm{Pic}^0(C) = \ker(\mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z})$.

In complex geometry, $\mathrm{Jac}(C) \cong H^{1,0}(C)^\vee / H_1(C,\mathbb{Z})$

$$\gamma \in H_1(C,\mathbb{Z}) \longmapsto \int_\gamma (\ \ ) \in H^{1,0}(C)^\vee$$

These Jacobians are complex tori – $\mathbb{C}^g/\mathbb{Z}^{2g}$ – & in fact alg. varieties.

[Even though Pic is not a (proj) variety, due to not being finite type]

The map (due to Abel) is:

$$P_1 \longmapsto \int_{P_0}^{P_1} \in H^{1,0}(C)^{\vee} \big/ H_1(C,\mathbb{Z})$$

where we integrate along paths $p_0 - p_1$; i.e. along δ s.t. $\partial\delta = p_1 - p_0$

This is well-defined after we mod out by $H_1(C,\mathbb{Z})$

Abel's Theorem $\Longrightarrow$ This is an embedding.

$Ch^0(C) = \{\sum n_i P_i,\ n_i \geqslant 0\} / \sim$ ← To be defined.

$Ch^0(C)_n = \{\sum n_i P_i \in Ch^0(C) \mid \sum n_i = n\}$

For points in $Ch^0(C)_n$, define a map:

$$\int_{nP_0}^{\sum n_i P_i} (\ ) \longmapsto \sum n_i \int_{P_0}^{P_i}$$

add in Jac(C), using group law

Fact

If $\sum n_i P_i$ is linearly equiv to $\sum m_j P_j$,

then $\displaystyle\int_{nP_0}^{\sum n_i P_i} = \int_{nP_0}^{\sum m_j P_j}$ ($\exists$ f meromorph. on C s.t. $\mathrm{div}(f) = \sum n_i P_i - \sum m_j P_j$)

This leads us to consider $\mathrm{Sym}^n C / \text{lin.equiv}$;

this has a well-defined map to the Jacobian.

(the one just defined).

If $n \geq g$, this is an isomorphism.

Griffiths-Abel-Jacobi map

$$Ch^p(X)_0 = \{ \sum n_i P_i \mid \sum n_i [P_i] = 0 \text{ in } H_{2p}(X, \mathbb{Z}) \}$$

If $\Gamma \in Ch^p(X)_0$, there exists $D \in Z_{2p+1}(X)$ s.t. $\partial D = \Gamma$. We define a f'n on $\left(H^{2p+1,0}(X) \oplus H^{2p,1}(X) \oplus \cdots \oplus H^{p+1,p} \right)^\vee$

$$\omega \longmapsto \int_D \omega$$

which will be well defined after we mod out by $H_{2p+1}(X, \mathbb{Z})$.

This is the "Griffiths intermediate Jacobian" (less algebraic than Weil's), which we will denote

$$J^p(X).$$

Chow Group

For X an alg. variety, p = dim of cycles we want to study.

$\Gamma \sim \Gamma'$ if there is a cycle $\Delta \subset X \times \mathbb{P}^1_Z$

s.t. $\Delta|_{Z=0} = \Gamma$, $\Delta|_{Z=\infty} = \Gamma'$

Then $Ch^p(X) = \{ \sum n_i V_i \mid \dim V_i = p, \ n_i \geq 0 \} / \sim$

w/ $\sim$ as above.

" $\partial \Delta = \Gamma' - \Gamma$ "

(Application of Griffiths int. Jac).

Clemens-Griffiths:

X = smooth cubic 3-fold $\subseteq \mathbb{P}^4$

$F(X)$ = Fano variety of lines

$= \{\ell \subseteq X \mid \ell \text{ is a line}\}$

$\dim F(X) = 2$

- Choose a line $\ell_0 \in F(X)$; we construct a map $F(X) \xrightarrow{\alpha} J'(X)$,

and the image of α generates all of $J'(X)$.

$$\left(\ell \mapsto \int_{\ell_0}^{\ell} \right)$$

One can show $J'(X) \not\cong \mathrm{Jac}(C)$ for any curve C, which shows X is not birational to $\mathbb{P}^3$.

The analog of $0 \to Jac(C) \to Pic(C) \to \mathbb{Z} \to 0$,

$$0 \longrightarrow J^p(X) \to H_D^{2p}(X, \mathbb{Z}) \longrightarrow H^{2p}(X, \mathbb{Z}) \longrightarrow 0$$

Deligne cohomology.

- If we use a quasi-projective X,

$H^*(X)$ has a "mixed" Hodge structure.

Consider a degeneration X_t, $t \in \Delta$, $X_t \to X_0$

$H^*(X_t) \circlearrowleft$ monodromy action has a "limiting mixed Hodge structure" e.g. $g(C) = 1$

$$H^2(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$$

$C_t \longrightarrow C_0 =$ 

limiting MHS: $H^{1,0} \longrightarrow H^0$

$H^{0,1} \longrightarrow H^2$

$$H_D^{2p}(X, \mathbb{Z}(n))$$

$\uparrow$

Tate Twist

Bloch

$$Ch^p(X, n) \xrightarrow{\exists ?} H_D^{2p}(X, \mathbb{Z}(n))$$

~~Motivation:~~

~~$\oplus\, Ch^p(X) \otimes \mathbb{Q}$~~

Motivation

$$\begin{tikzcd}
CH^p(X) \arrow[r] \arrow[d] & H^{2p}(X, \mathbb{Z}) \\
K(X) \arrow[ur] &
\end{tikzcd}$$

Quillen had defined higher algebraic K-theory.

2 variants of higher Chow groups:

(Bloch's original uses "algebraic geometer's simplex";

we use one w/ "algebraic geometer's cube".)

$$\square^n_{AG} = \left(\mathbb{P}^1 - \{1\}\right)^n$$

$$\left(\Delta^n_{AG} = \operatorname{Spec}\left(\mathbb{C}[t_1, \ldots, t_n] \big/ (1 - \Sigma t_j) \right) \cong \mathbb{C}^n \right)$$

Faces: set variables z_i to 0 or ∞.

$$CH^p(X) = Z^p(X) \big/ \operatorname{Im} \partial \, ,$$

Higher Chow Cycles

$$Z^p(X, n) = \{ \rho \in Z^p(X \times \square^n_{AG}) \}$$

$\partial_i : Z^p(X, n) \longrightarrow Z^p(X, n-1)$ using the i^{th} face.

Fact

$$\partial^2 = 0 \Rightarrow C^p(X,n) = \frac{\ker\left(Z^p(X,n) \to Z^p(X,n-1)\right)}{\operatorname{im}\left(Z^p(X,n+1) \to Z^p(X,n)\right)}$$

The paper cited in the beginning expresses the AJ map for homologically trivial cycles in terms of integrals:

$$\omega \in H^{k,\ell}(X)$$

$$X \times \square^n \qquad \rightsquigarrow \frac{1}{(2\pi i)^n}\int \pi_X^{*}\omega \wedge \pi_{\square}^{*}\left(\log z_1 \, d\log z_1 \dots\right)$$