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M. Romo

All order volume conjecture for closed hyperbolic 3-manifolds

Outline

- volume conjecture for $S^3 \setminus K$ (and its generalization)
- Chern-Simons theory $(G = SL(N, \mathbb{C}))$
- Dehn filling of $S^3 \setminus K \rightsquigarrow$ closed 3-manifolds
- $SL(2,\mathbb{C})$ Chern-Simons theory on closed 3-manifolds

all order V.C

Volume conjecture

$S^3 \setminus K$, hyperbolic

Jones polynomial: $\quad J(K, q) \in \mathbb{Z}[q, q^{-1}]$

defined by Skein relations:

$$q^2 J(\nearrow\!\!\!\!\nwarrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) J()()$$

and normalization $\quad J(\bigcirc) = q + q^{-1}$

Colored Jones polynomial: (reps of $SU(2)$)
$\hookrightarrow N \in \mathbb{Z}_{\geq 1}$

$$J_{\oplus R_i}(K,q) = \sum_i J_{R_i}(K,q) \quad \text{etc.}$$

$$J_N(K,q) \in \mathbb{Z}[q,q^{-1}]$$

$$V_N(K,q) := \frac{J_N(K,q)}{J_N(\bigcirc)}$$

<u>Volume conjecture</u> (for $S^3 \setminus K$)

(Kashaev, Murakami^2)

hyperbolic volume

$$2\pi \lim_{N\to\infty} \frac{\log\left[V_N(K, q=e^{\pi i/N})\right]}{N} = \mathrm{Vol}(M) + i\,CS(M)$$

Take $q = e^{\pi i/k}$ $\quad k\to\infty$, $N\to\infty$ with $u = i\pi N/k$ fixed.

Then $V_N(K,q)$

$$\sim e^{-\frac{1}{4\hbar}\left(\mathrm{Vol}(M;u) + i\,CS(M;u)\right) - \frac{3}{2}\log\hbar + \sum_{n\geq 1} S_n(u)\hbar^{n-1}}$$

$\hbar = i\pi/k$.

Chern-Simons theory

Data: gauge group $G = SL(2,\mathbb{C})$; 3-manifold M

principal G-bundle $\begin{array}{c} E_G \\ \downarrow \\ M. \end{array}$ (trivial)

Connection $\mathcal{A} \in \Omega^1(M, \mathfrak{g})$

$$CS(\mathcal{A}) = \int_M \mathrm{Tr}\left(\mathcal{A}\, d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$$

$$S_{CS} := \frac{t}{8\pi} CS[\mathcal{A}] + \frac{\tilde{t}}{2\pi} CS[\bar{\mathcal{A}}]$$

$t, \tilde{t} \in \mathbb{C}$.

$$Z_G(M) = \int_{\mathrm{Conn}(E_G)/\mathcal{G}} D\mathcal{A}\, D\bar{\mathcal{A}}\, e^{iS_{CS}}$$

To get a gauge-invariant thing, $t = k + is$ $\quad k \in \mathbb{Z}$

$\tilde{t} = k - is$ $\quad s \in \mathbb{C}$.

Compact $G = SU(2)$

$$Z_{SU(2)} = \int D\mathcal{A}\, e^{ik/4\pi\, CS(\mathcal{A})}$$

Witten '98:

$$Z_{SU(2)}(S^3 \setminus K) = \int D\mathcal{A}\, e^{ik/4\pi\, CS(\mathcal{A})}\, W_K^R(\mathcal{A})$$

where $W_K^R(\mathcal{A}) = \mathrm{Tr}_R(\mathrm{Hol}_K(\mathcal{A}))$

This is conjectured to be $J_R(K; q = e^{\pi i/k})$

<u>Gukov '03</u>

$Z_{SL(2,\mathbb{C})}(S^3 \setminus K)$

impose boundary conditions: $\mathrm{Hol}_{\gamma_m}(\mathcal{A}) \sim \begin{pmatrix} e^u & \\ & e^{-u} \end{pmatrix}$

where γ_m = meridian cycle

Do a perturbation expansion around a flat connection α = saddle point

$$Z^{(\alpha)}_{SL(2,\mathbb{C})}(S^3 \setminus K) \overset{(\text{as } \hbar \to 0)}{\sim} e^{\sum_{m \geq -1} \hbar^n S_m^{(\alpha)}(u) + \frac{3}{2}\log \hbar}$$

$\hbar = 2\pi i b^2$, $k = 1$, $s = -i(1-b^2)/(1+b^2)$

<u>cf Gukov-Dimofte-Zagier-Lenells '09</u>

If we take $u = i\pi$, we get a generalized version of the volume conjecture.

Volume conjecture for closed hyperbolic 3-manifolds

$\partial M = \phi$

Reshetikhin-Turaev invariant [?]:

based on quantum group techniques

$U_q(\underset{\sim}{sl_2})$

$$\tau_r^{SU(2)}(M) \in \mathbb{Z}[q, q^{-1}], \quad q = e^{\pi i/r}$$

$r \in \mathbb{Z}_{\geq 3}$

Asymptotic expansion conjecture (Witten '89)

$$\tau_r^{SU(2)}(M) \xrightarrow{r \to \infty} \Gamma^{\#}$$

(Lickorish, Kirby-Melvin)

$$\tau_r^{SO(3)}(M) \in \mathbb{Z}[q, q^{-1}] \qquad q = e^{(?)\pi i/r}$$

r odd

$r \in \mathbb{Z}_{\geq 3}$

Chen-Yang '15

(Numerically) Conjecture

$$2\pi \lim_{\substack{r \to \infty \\ r \text{ odd}}} \log(\tau_r^{SO(3)}(M))/r = \mathrm{Vol}(M) + i\, CS(M)$$

Moreover, $\log \tau_r^{SO(3)} \xrightarrow{r\to\infty} \sum_{n\geq -1} \hbar^n S_n(M) \Big|_{\hbar = \frac{2\pi i}{r}}$

- which 3-manifolds?

$$M = (S^3 \setminus K)_{p\gamma_m + q\gamma_\ell} = (p,q) \text{ Dehn filling s.t. } M \text{ is hyperbolic}$$

(To construct this, glue $D^2 \times S^1$ to $S^3 \setminus K$ s.t. $p\gamma_m + q\gamma_\ell$ is identified with the contractible cycle in $D^2 \times S^1$.)

$$\tau_r^{SO(3)}((S^3 \setminus K)_{p\gamma_m + q\gamma_\ell}) = F(r) \sum_{N=0}^{r-2} a_{N,r} J_{N+1}(K; e^{2\pi i/r})$$

<u>What is the physics?</u>

$$Z^{(k=1,b)}_{SL(2,\mathbb{C})}(S^3 \setminus K; u) = \langle u | S^3 \setminus K \rangle$$

$$Z_b(M) := Z^{(k=1,b)}_{SL(2,\mathbb{C})}((S^3 \setminus K)_{p\gamma_m + q\gamma_\ell})$$

$$= \langle D^2\times S^1 \,|\, \hat{\varphi} \,|\, S^3\setminus K\rangle = \int dx \,\langle D^2\times S^1 | x\rangle \langle x; \varphi, S^3\setminus K\rangle$$

D. Gang ...

$\varphi \in SL(2,\mathbb{Z})$ frame.

$\hat{\varphi}$ = operator

$$Z^{(k=1,\hbar)}_{SL(2,\mathbb{C})}(S^3\setminus K, u)$$

quadratic

$$\sim \int \prod_{i=1}^{L} \frac{dz_i}{\sqrt{2\pi\hbar}}\, e^{\frac{1}{\hbar} Q(z,u,A,B,C,D)} \prod_{i=1}^{T} \psi_\hbar(z_i)$$

Quantum dilogarithm

$$S^3\setminus K = \bigcup_{i=1}^{J} \Delta_i \quad \text{(ideal triangulation)}$$

A, B, C, D (Neumann-Zagier)

complete hyperbolic flat connection

$$Z_\hbar^{(\alpha)}(M) \sim e^{\sum S^{(n)}(M)\hbar^n}$$

<u>Conjecture</u> (D Gang, MR, Masahito Yamazaki '17)

$$Z_\hbar^{(\alpha)}(M) = e^{\sum_{m\geq -1} S(M)\hbar^m}$$

from

$$\tau_r^{SO(3)}(M)$$

(answer to question)

$$\langle (S^1 \times D^2)(x) \rangle \sim e^{\# x^2 + \# x} \sinh\left(\frac{x}{q}\right) \sinh(xq)$$

(answer to question 2)

<u>Conj</u>

$$e^{-2 S_1^{hyp}(M)} = \mathrm{Tor}\left((S^3 \setminus K)_{px + q p} \right)$$

Radermeister torsion