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2+1 Dimensional Gravity as an Exactly Solvable System (Witten '88)

Conflicting statements?

1) 3D gravity is "trivial" (Better: topological)

(M, g) $\qquad R_{ij} = 2\Lambda g_{ij}$, $\Lambda \in \mathbb{R}$

$\Lambda = 0$: locally Minkowski

$\Lambda > 0$: locally dS

$\Lambda < 0$: locally AdS

2) 3D gravity is non-renormalizable

$$I_{\text{Hilbert}}(g) = \frac{1}{16\pi G} \int (R - 2\Lambda)\, d\mathrm{Vol}_g$$

Newton's constant

$$[G] = \text{mass}^{-1}$$

$U \subseteq M$, $\dim M = d$.

$e_{(a)}$, $a = 1, \ldots, d$. $\qquad a, b, c, \ldots$

∂_i $\qquad i = 1, \ldots, d$ $\qquad i, j, k, \ldots$

$\partial_i = e_i{}^a e_{(a)}$: $e_i{}^a =$ "vielbein"

$e_{(a)} = e^i{}_a \partial_i$

$g(e_{(a)}, e_{(b)}) = \eta_{ab}$ $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$g_{ij} e^i{}_a e^j{}_b = \eta_{ab}$

$\nabla V = (\nabla_i V^a) dx^i \otimes e_a$

$\nabla_i V^a = \partial_i V^a + \omega_i{}^a{}_b V^b$

$\omega =$ "spin connection"

$R_{ij}{}^a{}_b = \partial_i \omega_j{}^a{}_b - \partial_j \omega_i{}^a{}_b + [\omega_i, \omega_j]^a{}_b$

$\omega_i \in \mathfrak{so}(d-1,1)$

$e : TU \to U \times \mathbb{R}^d$

$\omega = SO(d-1,1)$ connection on $U \times \mathbb{R}^d$

$M =$ smooth manifold, no a priori metric

$e : TM \to V$

$\eta =$ Lorentz metric on V

$\omega =$ a metric-compatible $SO(d-1,1)$ connection

Then can pull back to a metric g on M

New variables for gravity in 2+1 gravity

$g \longleftrightarrow$ e veilbein
ω spin connection

$R = d\omega + \omega \wedge \omega$

$$I_{Hilbert}(g) = \int R \sqrt{|\det g|}\, d^3x = \int_M e \wedge R$$

$$= \int \varepsilon_{abc}\, e_i^{\ a} \left(\partial_j \omega_k^{\ bc} - \partial_k \omega_j^{\ bc} + [\omega_j, \omega_k]^{bc}\right) dx^i \wedge dx^j \wedge dx^k$$

with EOM:
for ω: $\nabla_i e_j^{\ a} - \nabla_j e_i^{\ a} = 0$ (torsion-free)

for e: $e_a^{\ k} R_{ik\ b}^{\ \ a} = 0$ (Ricci-flat)

(Note: $\Lambda = 0$)

Gravity as Chern-Simons

$ISO(2,1)$ = Poincaré group in 2+1 dim

$iso(2,1)$ = Poincaré algebra

P^a, J^{ab} $\quad a, b = 0, 1, 2$

$$J^a := \frac{1}{2} \varepsilon^{abc} J_{bc}$$

$$[J_a, J_b] = \varepsilon_{abc} J^c$$

$$[J_a, P_b] = \varepsilon_{abc} P^c$$

$$[P_a, P_b] = 0$$

Ad-invariant inner product (trace) on iso(2,1)

$$\langle J_a, P_b \rangle = \delta_{ab} \text{ ; all others } 0.$$

Define an ISO(2,1) gauge field

$$A_i = e_i{}^a P_a + \omega_i{}^a P_a$$

where $\omega_i{}^a = \frac{1}{2} \varepsilon^a{}_{bc}\, \omega_i{}^{bc}$

$$I_{CS} = \frac{1}{2}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge [A \wedge A]\right)$$

$$= \int_M e_{ia}\left(\partial_j \omega_k{}^a - \partial_k \omega_j{}^a + \varepsilon_{abc}\, \omega_j{}^b \omega_k{}^c\right) dx^i \wedge dx^j \wedge dx^k$$

$$= \int_M e \wedge R$$ (= Hilbert action rewritten in new variables)

<u>CS</u>

Gauge symmetries

<u>Gravity</u>

Diffeo

Local Lorentz

Canonical quantization

Phase space: space of classical solution (of EOM) modulo symmetries

impose constraints:

e.g, $A_0 = 0$

Alternatively: Space of solution to constraint equation, modulo residual symmetries

$M = \Sigma \times \mathbb{R}$, Σ = closed orientable $g \geq 2$

$$I = -2\int dt \int_\Sigma \varepsilon^{ij} \frac{d\omega_j{}^a}{dt} e_{ia} + \cdots$$

$$p\dot{q}, \quad \{q, p\} = 1$$

$$\{\omega_i{}^a(x), e_j{}^b(y)\} = \frac{1}{2}\varepsilon_{ij}\eta^{ab}\delta^{(2)}(x-y)$$

$$0 = \frac{\delta I}{\delta e_0{}^a} = \varepsilon^{ij}(\partial_i \omega_j{}^a - \partial_j \omega_i{}^a + \varepsilon^{a}{}_{bc}\omega_i{}^b\omega_j{}^c)$$

$$0 = \frac{\delta I}{\delta \omega_0{}^a} = \cdots$$

$e_i{}^a P_a + \omega_i{}^a J_a$ ISO(2,1) connection on Σ

$\to$ flat!

$\mathcal{M}$ = flat $ISO(2,1)$ connections on Σ modulo gauge transformations

Configuration space

$\delta e \sim e + \omega$ $\longrightarrow$ ω should be coordinates,

$\delta \omega \sim \omega$ $\quad$ e should be momenta

ω = $SO(2,1)$ connection

$\mathcal{N}$ = moduli space of flat $SO(2,1)$ connections / gauge transformation

$\mathcal{M} = T^*\mathcal{N}$

$\mathcal{N}$ has connected components classified by

$n \in \mathbb{Z}$, $|n| \leq 2g-2$ (Euler class of bundle)

$\mathcal{N}' :=$ component with maximum Euler class $2g-2$

= configuration space

$T^*\mathcal{N}'$

$\mathcal{N}$ = moduli space of flat $SO(2,1)$ connection

= moduli space of homomorphisms $\pi_1(\Sigma) \to SO(2,1)$ modulo conjugation

$\mathcal{W}' = \pi_1(\Sigma) \xrightarrow{\varphi} SO(2,1)$ s.t. $\ker\varphi = 0$
and $\operatorname{im}\varphi$ is discrete

$g \geq 2$: $\Sigma = \mathbb{H}^2 / \operatorname{im}\varphi$ $\qquad$ $\mathbb{H}^2$ = upper half plane

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$\mathcal{H} = L^2(\mathcal{W}')$

$\psi \in L^2(\mathcal{W}')$

$\pi_1(\Sigma) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$

- $\psi = \psi(u_1, \ldots, u_g, v_1, \ldots, v_g)$
  s.t. $[u_1, v_1] \cdots [u_g, v_g] = \mathbb{1}$
- ψ invariant under conjugation
- $u_1, \ldots, u_g, v_1, \ldots, v_g$ define $\pi_1(\Sigma) \to SO(2,1)$
  with no kernel, discrete image
- $\int |\psi|^2 \, du_1 \cdots du_g \, dv_1 \cdots dv_g < \infty$