K3 Surfaces ~ Lecture 3

$X =$ K3 surface, Kählerian

Choose an isomorphism $\varphi : H^2(X;\mathbb{Z}) \longrightarrow \Lambda^{3,19} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

X determines a subspace $\underset{\mathbb{C}\cdot\omega}{\overset{\shortparallel}{H^{2,0}(X)}} \subseteq H^2(X,\mathbb{C})$

$\langle \omega, \bar{\omega} \rangle^{\perp} \subseteq H^2(X,\mathbb{C})$; $\langle \omega, \bar{\omega} \rangle^{\perp} = H^{1,1}(X,\mathbb{C})$

$$H^{1,1}(X,\mathbb{R}) = \langle \mathrm{Re}\,\omega, \mathrm{Im}\,\omega \rangle^{\perp} \text{ in } H^2(X,\mathbb{R})$$

Facts

- $H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z}) = \mathrm{Pic}(X)$

  i.e. $\forall\, \alpha \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$, $\alpha = c_1(\mathcal{L}) = c_1(\mathcal{O}(D))$

  $(\alpha = [D])$

- (Riemann-Roch):

  $$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \frac{D^2 - K\cdot D}{2} \qquad (K = K_X) \qquad ([K_X] = 0 \text{ for K3's})$$

  $$= 2 + \frac{D^2}{2}$$

  $$\Rightarrow h^0(\mathcal{O}) - h^1(\mathcal{O}) + h^2(\mathcal{O})$$
  $$\underset{1}{\shortparallel} \qquad \underset{0}{\shortparallel} \qquad \underset{1}{\shortparallel}$$

- If C is an irreducible alg. curve on X of genus g, then

  $2g - 2 = K_C = (K_X + C).C = C\cdot C = C^2 \Rightarrow \chi(\mathcal{O}(C)) = g+1$

- Fact 4

C irreducible $\Rightarrow h^1(\mathcal{O}(C)) = 0$

From:

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

we get, in the l.e.s. in cohomology

$$H^0(\mathcal{O}_X) \xrightarrow{\sim} H^0(\mathcal{O}_C) \to H^1(\mathcal{O}_X(-C)) \hookrightarrow \underset{\overset{\shortparallel}{0}}{H^1(\mathcal{O}_X)}$$

$$\Rightarrow H^1(\mathcal{O}_X(-C)) = 0$$

- Fact 3.5

Serre duality says $H^i(\mathcal{O}(L)) \simeq H^{2-i}(\mathcal{O}(K_X - L))^* = H^{2-i}(\mathcal{O}(-C))^*$

$$\Rightarrow H^1(\mathcal{O}_X(C)) = 0$$

- $h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) + h^0(\mathcal{O}(-D)) = 2 + \frac{D^2}{2}$

$\Rightarrow$ If $D \cdot D \geq -2$, then $h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(-D)) > 0$

$\Rightarrow$ Either D or $-D$ is effective; if $D \neq 0$, exactly one is effective.

If C is irreducible, $h^0(\mathcal{O}(C)) = \chi(\mathcal{O}(C)) = g+1$.

$H^2(X,\mathbb{Z}) \supset \{[D] : h^0(\mathcal{O}(D)) > 0\} \supseteq \{[C] : h^0(\mathcal{O}(C)) > 0 \text{ \& } C \text{ irred}\}$ (3)

$$\cap\!|$$

$$\{\alpha \in H^2(X,\mathbb{Z}),\ \alpha^2 \geq -2\}$$

- $\omega \in H^2(X,\mathbb{R})$ is Kähler

$$\Longleftrightarrow$$

$$\langle \omega, C \rangle > 0 \quad \forall\ C \in N^+(X)$$

- $\mathrm{Pic}(X)^\perp \subseteq H^2(X,\mathbb{R})$; If $\mathrm{Pic}(X) = \{0\}$, 20 complex parameters for the K3

(More generally, if $\mathrm{rk}\,\mathrm{Pic}\,X = \rho$, $20-\rho$ parameters for K3)

Rational Curves

$C \in X$ curve, then $C^2 = -2 \Leftrightarrow$ (C irreducible, rational curve)

Choose D so that $D^2 > 0$. Look at

$$\{C \mid D \cdot C = 0,\ C \text{ effective}\}$$

~~Note: Signature of $\langle,\rangle$ on $\mathrm{Pic}(X)$ is $(1,\rho-1)$~~

3 cases for signatures

- $\mathrm{Pic}(X)$ has signature $(1, p-1)$
- $\mathrm{Pic}(X)$ has signature $(0, p-1, 1)$
- $\mathrm{Pic}(X)$ has signature $(0, p)$

Then $\{C \mid D\cdot C = 0,\ C \text{ effective}\}$ is negative definite.

Let $C_1, \dots, C_k$ generate the lattice

$L = \{C \mid D\cdot C = 0,\ C \text{ effective}\}$

$C_i^2 = -2, \quad C_i \cdot C_j \geq 0 \quad \text{if } i \neq j.$

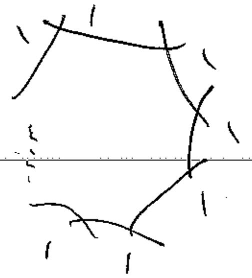
Classification : ADE

- If $C_i \cdot C_j \geq 2 \quad \forall\, i,j$ then:

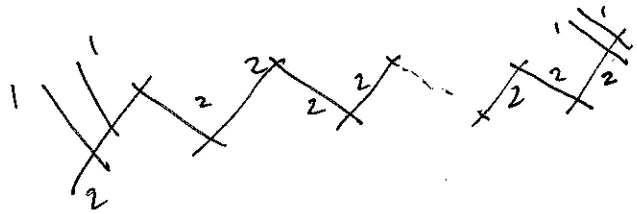
$$(C_i + C_j)^2 = -2 + 2\cdot C_i \cdot C_j - 2 \geq 0$$

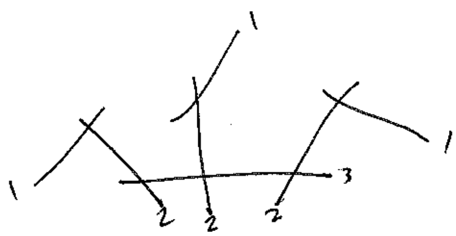
$*$

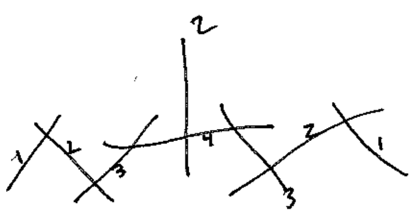
$\therefore$ two distinct curves can meet at at most 1 point.

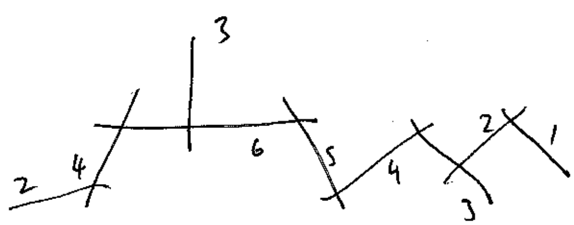


$$(C_0 + \cdots + C_{n-1})\cdot C_i = 1-2+1=0$$



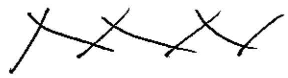




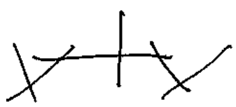


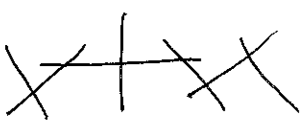
These curve configurations are the ones we wish to avoid.

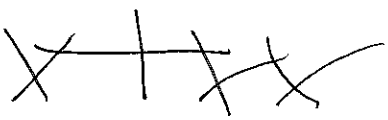
If we avoid those graphs, ~~and graphs~~ and any containing ~~them, we're left w/:~~

A_n  n

D_n  n

E_6 

E_7 

E_8 

Each of these has a reflection group associated:

If C is a rational curve, then:

$$D \mapsto D + (D, C) \cdot C$$

preserves inner products;

$$C \mapsto C + -2C = -C$$

If $D \cdot C = 1$, then $D \mapsto D + C$ $\quad (D + C)^2 = D^2$

$D \cdot C = -1 \Rightarrow D \mapsto D - C$

$R = \{D \in L \mid D^2 = -2\}$ decomposes as

$$R = R_+ \cup R_-$$

where $R_+ = \{D : D\cdot C > 0\}$ (effective) $\quad R_- = \{D : D\cdot C < 0\}$

This decomposition corresponds to choosing a set of positive roots (if we compare to Lie algebra story).

Say ω is almost Kähler, i.e. $\omega^2 > 0$ & $\omega \in \overline{\{\text{Kähler}\}}$.

$$\varphi : H^2(X, \mathbb{Z}) \longrightarrow \Lambda^{3,19}$$

$$\omega \longmapsto \varphi(\omega) \in \overline{\{\text{given positive cone}\}} \cup \text{(translates by Weyl of } \omega^{\perp}\text{)}$$

$\omega \leadsto$ Kähler class on $\bar{X}$ = result of blowing down C_i s.t. $(\omega, C_i) = 0$.

The complex singularities we get from blowing down are all of the form $\mathbb{C}^2/\Gamma$ for $\Gamma \in SL_2(\mathbb{C})$ a finite subgroup.

$A_n \leadsto \mathbb{Z}/n+1$

$D_n \leadsto BD_{f(n)}$ — $f(n) = 8n+3$ or $4n+3$ or something similar.

$E_6 \leadsto BT$

$E_7 \leadsto BO$

$E_8 \leadsto BI$

Kummer

For which k do there exist a quartic surface $\bar{X}$ in $\mathbb{P}^3$ which has precisely k ordinary double points? (A_1 singularity)

$\mathbb{C}^2/\mathbb{Z}/2 \rightsquigarrow (u,v) \mapsto (-u,-v) \rightsquigarrow (\mathbb{C}^2)^{\mathbb{Z}/2} = \mathbb{C}[u^2, uv, v^2] = \mathbb{C}[x,y,z]/xz-y^2$.

Resolving singularities $X \longrightarrow \bar{X}$, we get curves $C_1, \ldots, C_k$,

$C_i \cdot C_j = -2\delta_{ij}$ $\quad D = \pi^* H \Rightarrow D^2 = 4,\ D \cdot C_i = 0$

Our matrix is $\begin{pmatrix} -2 & & 0 \\ & -2 & \\ 0 & & \ddots \\ & & & -2 \end{pmatrix}$.

Look at $\mathbb{Z}\langle C_1, \ldots, C_k\rangle \overset{\varphi}{\subseteq} \Lambda^{3,19}$.

$\varphi(\underbrace{\mathbb{Z}\langle C_1, \ldots, C_k\rangle}_{M}) = \Lambda^{3,19}$; $\operatorname{rk} M = k$, $\quad \operatorname{rk} M^\perp = 22-k$.

This tells us automatically that $k \le 22$ (and in fact ≤ 19, since the lattice is neg. def).

$M \oplus M^\perp \subseteq \Lambda^{3,19}$

Nikulin's Discriminant Form:

$q : M^\vee / M \longrightarrow \mathbb{Q}/\mathbb{Z}$

$M^\vee = \operatorname{Hom}(M, \mathbb{Z}) \subseteq M \otimes \mathbb{Q}$

$q : M^\vee/M \to \mathbb{Q}/\mathbb{Z}$; $q^\perp : (M^\perp)^\vee/M^\perp \to \mathbb{Q}/\mathbb{Z}$

$$q = -q^\perp$$

provided that $\Lambda^{3,19}/M$ is torsion-free, i.e. M primitive.

Observe

The number of generators of $M^\vee/M$ is $\leq \operatorname{rk} M$.

$\Rightarrow$ # gens $(M^\perp)^\vee/M^\perp \leq 22-k$

$L^\vee/L = (\mathbb{Z}/2)^{6k} \Rightarrow k \leq 11$

But 11 is smaller than 16...

$L \hookrightarrow \Lambda^{3,19}$; If cokernel has torsion, then

$\exists \sum n_i C_i \in \Lambda^{3,19}$, $n_j \in \mathbb{Q} \setminus \mathbb{Z}$ for some j.

Given $C_1, \dots, C_k$, what can we say about: (10)

$$\left\{ \text{subsets } I \subseteq \{1, \dots, k\} \;\middle|\; \frac{1}{|I|} \sum_{i \in I} C_i \in \Lambda^{3,19} \right\} ?$$

(Nikulin):

$$\left(\frac{1}{2} \sum C_i\right)^2 = \frac{1}{4}(-2)\#I \in 2\mathbb{Z} \Rightarrow \#I \subseteq 4\mathbb{Z}$$

But $\#I = 4$ is not possible, which would force

$\frac{1}{2}(C_1 + \dots + C_4)$ is effective;

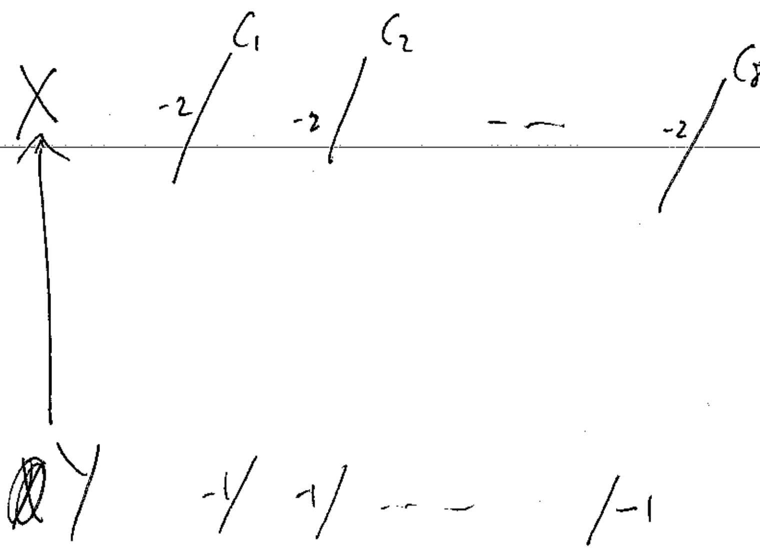
we assumed that we don't have a D_n graph (isolated curves)

$\# 8$ intersecting; $\# 12$ ~~[illegible]~~ bad / don't work; $\# I = 16$

$$M^\vee / M \cong (\mathbb{Z}/2\mathbb{Z}) \quad ; \text{ if } M \subseteq N, \quad N/M \cong \mathbb{Z}/2^{\oplus \ell}$$

~~[illegible]~~ $k - 2\ell \leq 22 - k$

$$k \leq 11 + \ell \Rightarrow k \overset{?}{\leq} 11$$



sum divisible by 2.
$(\sum C_i = 2D)$

$\bar{Y}$ is a nonsingular K3

$9 \circlearrowleft i$

$\iota^* \omega = \omega$; ι has 8 f.p.

$k = 1, \dots, 11 \Rightarrow$ no extra class

$k = 8, \dots, 12 \Rightarrow$ 1 extra class

$k = 12, 13 \Rightarrow \ell = 2$

$k = 14 \Rightarrow \ell = 3$

$k = 15 \Rightarrow \ell = 4$

$k = 16 \Rightarrow \ell = 5$

$\ell = 2$

$\mathbb{P}^2_{\mathbb{F}_2}$:

identity

$\mathbb{P}^1_{\mathbb{F}_2}$

3 pts

For 15, $\mathbb{P}^3_{\mathbb{F}_2} \geq \mathbb{P}^2_{\mathbb{F}_2}$.

For $k=12, 13 \longrightarrow Y' \circlearrowleft (\mathbb{Z}/2)^2$

$k=14 \longrightarrow Y'' \circlearrowleft (\mathbb{Z}/2)^3$

$k=15 \longrightarrow Y''' \circlearrowleft (\mathbb{Z}/2)^4$

$k=16 \qquad T^4 \circlearrowleft \mathbb{Z}/2$