Introduction to K3 Surfaces (Lecture 1)

0) Long term goal: Study recent constructions of manifolds with holonomy G_2

* A recent construction exploits geometry of K3's
* Many such manifolds are K3 fibrations.

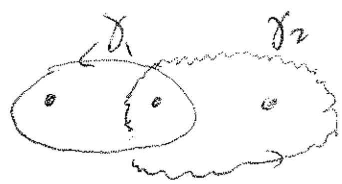
Elliptic Curves

1) $E \subseteq \mathbb{P}^N$, $\dim E = 1$, nonsingular $/k$ s.t. $\omega_E \cong \mathcal{O}_E$ (i.e. $\exists$ nowhere vanishing 1-form on E)

If $\exists P \in E$, & char $k \neq 2, 3$, then E can be described as:

$$y^2 z = x^3 + Axz^2 + Bz^3 \quad ; \text{ the 1-form is } \frac{dx}{y} = \frac{dx}{\sqrt{x^3 + Ax + B}}$$

2) If $k = \mathbb{C}$, we can use $\int_\gamma \frac{dx}{\sqrt{x^3 + Ax + B}}$ to study the elliptic curve.



$$(\omega_1, \omega_2) = \left(\int_{\gamma_1} \frac{dx}{\sqrt{x^3 + Ax + B}}, \int_{\gamma_2} \frac{dx}{\sqrt{x^3 + Ax + B}} \right)$$

The point is that $z := \int \frac{dx}{\sqrt{x^3 + Ax + B}} \in \mathbb{C}/\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

(Note: we can choose ω_1, ω_2 s.t. $\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$)

gives an isomorphism between E & $\mathbb{C}/\Lambda$.

3) Flat metric on $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ inherited from $\mathbb{C} = \mathbb{R}^2$.

Now we go to K3 surfaces:

1) Non singular projective variety X of dimension 2

s.t. $\omega_X \cong \mathcal{O}_X$ (i.e. $\exists$ nowhere vanishing 2-form) and

every 1-form is $\equiv 0$.

Examples

Let $X = V(F_1, \dots, F_m) \subseteq \mathbb{P}_k^{2+m}$. Then $\omega_{\mathbb{P}^{2+m}} \cong \mathcal{O}(-3-m)$

$\deg F_j = d_j$

The adjunction formula says $\omega_{V(f)} = \omega_Y|_{V(f)} \otimes N_{V(f)/Y}$

$\Rightarrow \omega_X = \mathcal{O}(-3-m+d_1+\dots+d_m)$

$\Rightarrow d_1 + \cdots + d_m = m + 3$

If $d_i = 1$, we're just changing the dimension of proj. space, so we may as well assume $d_i \geq 2$ $\forall i$.

Then the remaining possibilities are

$d_1 = 4$ inside $\mathbb{P}^3$ (degree 4).

$d_1 = 2, d_2 = 3$, inside $\mathbb{P}^4$ (degree 6)

$d_1 = d_2 = d_3 = 2$ inside $\mathbb{P}^5$ (degree 8)

⋮

$\exists$ X of degree $2g-2$ in $\mathbb{P}^g$

(Note: $X \cap H$ is a curve of genus g)

$\dim \{X \text{ of deg } 2g-2 \text{ in } \mathbb{P}^g\} = 19$

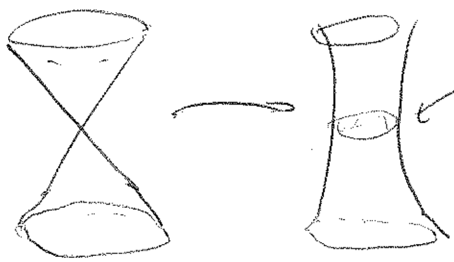
We can also look at a 2-1 cover of $\mathbb{P}^2_k$ (char $k \neq 2$) branched in a plane sextic, e.g.

$$t^2 = x^6 + y^6 + z^6$$

Example 2

$\mathbb{C}^2/(\mathbb{Z}^4 \rtimes \{\pm 1\}) \rightsquigarrow$ Gives a "singular" K3 surface

There are 16 (A_1) singularities $\mathbb{C}^2/\{\pm 1\}$.



C_α, $C_\alpha^2 = -2$

we get $\Rightarrow$ $Bl_{16}(\mathbb{C}^2/\mathbb{Z}^4 \rtimes \{\pm 1\})$ = "Kummer surface"

The Kummer surface we get after blowing up 16 points is an example of a K3.

Historical Observation

The K3 surface we got in example 2 (the Kummer) ~~that~~ was observed to have the same Betti numbers of a quartic in $\mathbb{P}^3$, which prompted further study.

The Betti numbers are:

$$b_0 = 1 = b_4 \; ; \quad b_1 = b_3 = 0 \; ; \quad b_2 = 22$$

This was pursued by A. Weil and others.

2.) Over $\mathbb{C}$: X is a compact complex surface

Y. Siu (1970's): Every K3 is Kähler.

X = Kähler surface, which means that X has a Hermitian Riemannian metric.

$$g_{\alpha\bar{\beta}} \, dz^\alpha \cdot d\bar{z}^\beta$$

2-form $\omega = \frac{i}{2\pi} g_{\alpha\bar{\beta}} \, dz^\alpha \wedge d\bar{z}^\beta$

Kahler means $d\omega = 0$.

locally $\omega = \frac{i}{2\pi} \partial\bar{\partial} \log K$

"Kahler Potential"

Our X, to recap, satisfies:

- $\dim_{\mathbb{C}} X = 2$
- $\exists$ nowhere vanishing holo. 2-form
- every holomorphic 1-form is $\equiv 0$
- is Kähler

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X) \qquad \left(H^{p,q}_{\bar\partial}(X) = \overline{H^{q,p}_{\bar\partial}(X)} \right)$$

Other facts

- Dualizing sheaf of X being trivial is equivalent to the existence of a global holomorphic $(2,0)$-form w/o 0's, which we denote Ω

$$b_2 = 1 + h_{11} + 1$$

1st Chern class

$$c_1(T_X) \in H^2(X, \mathbb{Z}) \qquad [Z(\Omega)] = c_1(T_X^{1,0})$$

holomorphic tangent bundle

$$c_1^2 = 0$$

Hirzebruch-R-R $\Rightarrow$ $\underset{\substack{\| \\ 0}}{c_1^2} + \underbrace{c_2}_{\chi(X)} = 12(1 - h^{10} + h^{20})$

Euler characteristic

$$12(1-0+1) = 24 \qquad b_2 = 22$$

$$\Downarrow$$

$$h_{11} = 20$$

The signature of the intersection form on H^2 is:

$$(1 + 2h^{20}, h^{11} - 1) = (3, 19)$$

We have the ~~signature~~ cup product pairing:

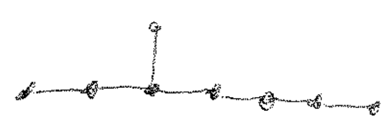
$$\cup : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

which is even (so $e \cup e \in 2\mathbb{Z}$ for all $e \in H^2$).

Fun fact:

K3 surfaces named after K2 the mountain, and also

Kähler, Kummer and Kaeser or Kodaira



$$H^2(X, \mathbb{Z}) \xrightarrow{\sim} \begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus \begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus \begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus -E_8 \oplus -E_8$$

<u>Note:</u> ~~for~~ elliptic curves:

$$H^1(E,\mathbb{Z}) \times H^1(E,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow \quad \det = \pm 1$$

Periods

- Choose basis $\gamma_1, \dots, \gamma_{22}$ of $H_2(X,\mathbb{Z})$
- Construct $X \to \mathbb{P}^{21}_{\mathbb{C}}$ by $p \mapsto \left[\int_{\gamma_1}\Omega, \dots, \int_{\gamma_{22}}\Omega\right] \in \mathbb{P}^{21}_{\mathbb{C}}$
- We know $(v,v)=0$ & $(v,\bar{v})>0$
  $\Omega\wedge\Omega \qquad \Omega\wedge\bar{\Omega}$

So the image of the period map is an open set

$U \subseteq V(\text{int form}) \subseteq \mathbb{P}^{21}_{\mathbb{C}}$

$\dim_{\mathbb{C}} U = 20$

3) Metric

Calabi conjectured & Yau proved

Given X, ω, $\exists!\ \omega'$ s.t. $Ric_{\omega'} \equiv 0$ and $[\omega'] = [\omega] \in H^2_{dR}(X)$

The holonomy of ω' is $SU(2) \subseteq SO(4)$.

Underlying Riemannian metric has an S^2 of compatible complex structures.

The dimension of the space of metrics is $40 + 20 - 1 - 2 = 57$.

g_{ij} = $SU(2)$-holonomy metric on underlying K3 4-manifold

$$\downarrow$$

$$\dim \left\{ \alpha \in \mathcal{H}^2(X, \mathbb{R}) \;\middle|\; \alpha \text{ harmonic},\ \star\alpha = \alpha \right\} = 3$$

$$\cap$$

$$H^2(X, \mathbb{R})$$

$$\dim O^+(3,22) / O(3)^* \times O(19) = 57$$