2/12/2016

Big Plan: Take elliptically fibered Calabi-Yau manifolds and understand good birational models, also good birational models of Jacobians!

$$\begin{array}{c} X \\ \downarrow \\ B \end{array} \Rightarrow \quad \begin{array}{c} \bar{X} \\ \downarrow \\ \bar{B} \end{array} \quad \begin{array}{c} \widetilde{J(X)} \\ \downarrow \\ \tilde{B} \end{array} \quad \begin{array}{c} W(X) \leftarrow \hat{W}(X) \\ \downarrow \\ \tilde{B} \end{array}$$

Birational geometry and minimal model program

$$B \xleftarrow{\text{blowup}} B_1 \xleftarrow{\text{blowup}} B_2 \leftarrow \cdots$$

Start with B, blow up many times

A good minimal model will satisfy $K_B \cdot C \geq 0$ $\forall$ curves C

We will blow up & then blow back down

Can find a minimal model for either B or X

$$K_X = \pi^*(K_B + \Lambda)$$

Λ — divisor with $\mathbb{Q}$-coefficients

Looking for curves C with $K_X \cdot C < 0$ & blow down

Variant

Look for curves C w/ $(K_B + \Lambda)C < 0$

and blow down.

In dimension ≥ 3, we may find that our minimal models

have singularities.

Key Properties that We Use

1) $X \to B$ is a flat family, i.e. all fibers

$\pi^{-1}(b)$ have dim 1

2) The set of singular fibers $\Delta \subset B$, as a starting point, is a divisor with normal crossings, i.e.

$$\Delta = \sum \Delta_j \qquad \text{each } p \in \bigcap_{j \in J(p)} \Delta_j$$

$$\Delta_{j_1} \cap \cdots \cap \Delta_{j_k} \text{ meet normally} \quad ((z_j = 0) = \Delta_j)$$

Will momentarily ignore Calabi Yau property (until we have our minimal model).

3) The J-invariant of the elliptic fibration is well-defined.

We'll lose 2, 3 after we start blowing up/down, possibly, but we insist on keeping 1.

Kodaira Classification & Tate variant

(How to resolve singularities in cod im 1 on B)

Kodaira

$$\begin{array}{c} X \\ \downarrow \pi \\ B \end{array} \qquad \dim X = 2, \quad \dim B = 1$$

Assume $b \in B$ has small neighborhood

X smooth, X does not contain

curves C s.t. $\pi(C) = b$

$C^2 = -1$, $K_X \cdot C = -1$

(any C could be blow down to a nonsingular point)

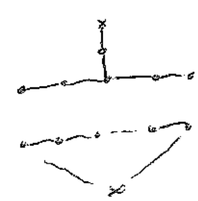
$\pi^{-1}(b) \in$ nice list

$\pi^{-1}(b) = \bigcup E_j$

If $\pi^{-1}(b)$ has at least two components, $E_j^2 = -2$

$\{E_j\}$ forms an affine Dynkin diagram

$\pi^{-1}(b)$ can have multiplicity m



mI_n, mI_0

We won't see the multiplicity in what follows...

(Weierstrass model)

Weierstrass model

$$\begin{tikzcd} X \arrow[d] \\ B \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

An genus 1 curve with point (section)

$$\Gamma(\mathcal{O}_X(n\sigma(B)))$$

$$\boxed{\text{All } \cancel{\text{fibers}} \text{ components of fibers of } \pi \text{ which do not meet the section } \sigma(B) \text{ are contracted to points.}}$$

$$y^2 = x^3 + fx + g \qquad (f \in \Gamma(\mathcal{L}^{\otimes 4}),\ g \in \Gamma(\mathcal{L}^{\otimes 6}))$$

$$\cap$$

$$\underline{\mathrm{Spec}}(\mathcal{O} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3})$$

$$(f,g) \mapsto (\lambda^4 f, \lambda^6 g)$$

OR

$$y^2 = x^3 + u x^2 + vx + w$$

$$u \in \Gamma(\mathcal{L}^{\otimes 2}),\ v \in \Gamma(\mathcal{L}^{\otimes 4}),\ w \in \Gamma(\mathcal{L}^{\otimes 6})$$

Note - $\Delta = \{4f^3 + 27g^2 = 0\}$

In the other form, $y^2 = x^3 + ux^2 + vx + w$

$$\Delta = \{4u^3w - u^2v^2 - 18uvw + 27w^2 + 4v^3 = 0\}$$

Select a curve $C \subset B$ such that C is a component of Δ. Analyze X at a generic point of C.

In local analytic coords, $C = \{z=0\}$.

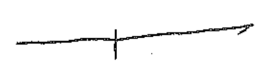
Case 1

C is not a component of Δ, fibers are nonsingular

$\therefore$ X smooth near C,

Case 2

$(z=0) \subseteq \Delta \Rightarrow z \mid \Delta$



By translating, $x \mapsto x+a$, we can put singular fiber at the origin (but now we have an eq'n w/ an x^2 term, i.e. in u, v, w coords).

$\therefore$ $z \mid w$, $z \mid v$; $I_n \hookrightarrow z \nmid u$

If $\mathrm{ord}_c(\Delta)=n \Rightarrow y^2=x^3+ux^2+z^{[n/2]}v_n x+z^n w_n$

The n determines what type of singularity we get.

$$y^2=ux^2+z^n$$

A_{n-1} singularity

$n-1$ curves

Blow up singularity

$$y^2=ux^2+z^n \longrightarrow \bar{y}^2=u\bar{x}^2+\bar{z}^{n-2}$$

$$\bar{y}^2=u\bar{x}^2$$

If u has a square root, $(\bar{y}-\sqrt{u}\,\bar{x})(\bar{y}+\sqrt{u}\,\bar{x})$ so we get 2 curves.

So we should figure out whether u is a square.

$$u=-9g/2f\,\big|_c$$

Stable means u is a square, not stable means u is not a square

Case 3

$z|u,\ z|v,\ z|w$ (so $z|f$ & $z|g$) so we go back to our Weierstrass model $y^2=x^3+fx+g$, which we can rewrite:

$$y^2=x^3+zf_1x+zg_1$$

Keep checking how much $z \mid f, g$, to get higher types.

II : $y^2 = x^3 + z f_1 x + z g_1$  (Case 3)

III : $y^2 = x^3 + z f_1 x + z^2 g_1$  A_1 (Case 4)

IV : $y^2 = x^3 + z^2 f_2 x + z^3 g_2$ 

A_2 (Case 5)

Case 6, 7

$z^2 \mid f$, $z^3 \mid g$

$\longrightarrow \operatorname{ord}(\Delta) = n+6$

These lead to I_n^*, associated to D_{n+4} singularity.

Case 8

$z^3 \mid f$, $z^4 \mid g$

[Cf Appendix B, 1109.0042, or Tate's paper from the 70's]

Next Time:

Codimension 2!

	$\mathrm{ord}_c(\Delta)$	$\mathrm{ord}_c(f)$	$\mathrm{ord}_c(g)$	
I_0	0	—	—	
mI_0	—	—	—	m 
If $z \mid uw$, then $z \nmid u$ I_n	n	0	0	 we get n E's, $E_j^2=-2$, $E_j \cdot E_{j+1}=1$
mI_n				m 
II	2	≥ 1	1	
III	3	1	≥ 2	
IV	4	≥ 2	2	
I_n^*	$n+6$	2	3	affine D_{n+4} 
IV^*	8	≥ 3	4	$\tilde{E}_6$
III^*	9	3	≥ 5	$\tilde{E}_7$
II^*	10	≥ 4	5	$\tilde{E}_8$