Elliptic Calabi-Yau Torsors

Application to F-theory: $/\mathbb{C}$ $\dim_{\mathbb{C}} X = d+1$

$\dim_{\mathbb{C}} B = d$

Elliptic CY Variety $\rightsquigarrow$ Weierstrass Model

$X \xrightarrow{\pi} B$ $\qquad y^2 = x^3 + fx + g$

$\pi^{-1}(b)$ = elliptic curve for general $b \in B$

$\in \mathcal{L}_1 \oplus \mathcal{L}_2 \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$

$\downarrow \qquad\qquad \downarrow$

$B \qquad\qquad B$

where f, g sections of line bundles on B

Used in physics for "F-theory": $10-2d$ dimensions

$S^1 \times \mathbb{R}^{8-2d,1}$ "M-theory on X"

If there are multiple $X_1, \dots, X_k$ with the same f, g, then F-theory has a discrete symmetry of order k.

Birational Geometry of X:

$K_X = 0$, $\omega_X \cong \mathcal{O}_X$, $\exists$ no-where vanishing holo $(d+1)$-form on X.

X should be a (Mori) minimal model:

- only 'mild' singularities are allowed
- $\forall$ alg curves $C \subset X$,

$$\deg \omega_X|_C \geq 0$$

$X \to B$

- π is a genus 1 fibration if

$g(\pi^{-1}(b)) = 1$ for general b.

$\Rightarrow$ an alg. curve of genus 1 over $K =$ [illegible]

- π is an elliptic fibration if $\overline{FL} = K(B)$ there is a 'rational section';

$S \subset X$ $\qquad \deg \mathcal{O}_{\pi^{-1}(b)}(S) = 1$

$\searrow \downarrow$

$\quad B$ $\qquad$ for b general.

$\Rightarrow S \to B$ is generally 1-1, onto,

$\Rightarrow$ an elliptic curve over K (genus 1 with point)

$K = \mathbb{Q}$ $\qquad y^2 = x^4 + 2$

genus 1 curve C

$V(-y^2 + x^4 + 2z^4) \subset \mathbb{P}^{(1,2,1)}$

has no $\mathbb{Q}$ rational point.

But this curve has a Jacobian $J(C)$, $y^2 = x^3 - 8x$

has several rational points

The group scheme $J(C)$ acts on C

$J(C) \times C \longrightarrow C$

$\Rightarrow$ C is a principal homogeneous space or torsor over $J(C)$.

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elliptic curve$/K$ $\longrightarrow$ Weierstrass Model

$P \in E$, $\mathcal{O}_E(P)$, $\mathcal{O}_E(nP)$

Riemann-Roch theorem $\Rightarrow$

$$\dim \Gamma(\mathcal{O}_E(nP)) = n-1+1 = n.$$

$\Gamma(\mathcal{O}_E(P)) \ni 1$ (trivializes $\mathcal{O}_E(P)$ on $E \setminus \{P\}$)

$\Gamma(\mathcal{O}_E(2P)) \ni 1^2, x$

$\Gamma(\mathcal{O}_E(3P)) \ni 1^3, 1 \cdot x, y$

$\Gamma(\mathcal{O}_E(4P)) \ni 1^4, 1^2 x, 1 \cdot y, x^2$

$\Gamma(\mathcal{O}_E(5P)) \ni 1^5, 1^3 x, 1^2 y, 1 x^2, xy$

$\Gamma(\mathcal{O}_E(6P)) \ni 1^6, 1^4 x, 1^3 y, 1^2 x^2, 1xy, x^3, y^2$

$\Rightarrow$

$$0 = -y^2 - a_1 xy - a_3 y + x^3 + a_2 x^2 + a_4 x + a_6$$

if char $K \neq 2, 3$ $\Rightarrow$

$$0 = (y - \tfrac{1}{2} a_1 x - \tfrac{1}{2} a_3)^2 + x^3 + (a_2 + \tfrac{1}{4} a_1^2) x^2 + (a_4 + \tfrac{1}{2} a_1 a_3) x + (a_6 + \tfrac{1}{4} a_3^2)$$

$$0 = -(y + \tfrac{1}{2} a_1 x + \tfrac{1}{2} a_3)^2 + (x + \tfrac{1}{3} a_2 + \tfrac{1}{12} a_1^2)^3 + f_4 (x + \tfrac{1}{3} a_2 + \tfrac{1}{12} a_1^2) + g_6$$

$$0 = -y^2 + x^3 + f_4 x + g_6$$

$$(x, y) \longmapsto (u^2 x, u^3 y)$$

$$u^{-6}((-u^3 y)^2 + (u^2 x)^3 + f_4 (u^2 x) + g_6)$$

$$= -y^2 + x^2 + u^{-4} f_4 x + u^{-6} g_6$$

$K \supset \mathcal{O}$ ($\mathcal{O} = \mathcal{O}_U$ for open $U \subset B$)

$f_4, g_6 \in K$

Choosing some $u^{-1} \in \mathcal{O}$

$u^{-4} f_4, u^{-6} g_6 \in \mathcal{O}$

WLOG, $f_4, g_6 \in \mathcal{O}$

If $\exists\, v \in \mathcal{O}$ s.t.

$$v^4 \mid f_4, \quad v^6 \mid g_6$$

we can change coords

$$(f_4, g_6) \longmapsto \left(\frac{f_4}{v^4}, \frac{g_6}{v^6}\right)$$

<u>Lemma</u> $\exists\, f_4, g_6$ s.t.

$$v^4 \mid f_4, \quad v^6 \mid g_6$$

$\Rightarrow v^4, v^6$ units in $\mathcal{O}$.

(Weierstrass minimal model

$$U \subset B$$

$$f_4, g_6 \in \mathcal{O}_U$$

$$V(-y^2 + x^3 + f_4 x + g_6) \subseteq \mathcal{O}_U \oplus \mathcal{O}_U \to U$$

$$\overbrace{V(-y^2 + x^3 + f_4 x z^4 + g_6 z^6)}^{W} \subseteq \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \to B$$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ B \end{array} \qquad \begin{array}{l} f_4 \in \Gamma(\mathcal{L}^{\otimes 4}) \\ g_6 \in \Gamma(\mathcal{L}^{\otimes 6}) \\ x \in \Gamma(\mathcal{L}^{\otimes 2}) \\ y \in \Gamma(\mathcal{L}^{\otimes 3}) \end{array}$$

$$\begin{tikzcd} W \arrow[d] & & X \arrow[ll, hook'] \arrow[d] \\ B & \text{is birational to} & B \end{tikzcd}$$

C = genus 1 curve over k

$J(C) = \{$ line bundle of degree 0 on $C\}$

$=$ Picard scheme of C

$[\mathcal{O}_C] \in J(C)$: data of elliptic curve

$$J(C) \times C \longrightarrow C$$

$$\mathcal{L} \qquad p$$

$$\mathcal{L} \otimes \mathcal{O}(p) \cong \mathcal{O}(q), \quad q \in C$$

$$\begin{array}{c} Y \\ \downarrow \\ B \end{array} \quad \text{Genus 1 fibration}$$

Modify Y (birationally) into a form which $J(Y)$ can be easily defined. (i.e. blowup base and fiber)

Also in this form,

$$\omega_Y = \pi^*(\omega_B(\Lambda))$$

Has $\mathbb{Q}$ coeff.

$$K_Y = \pi^*(K_B + \Lambda)$$

$$12 K_Y = \pi^*(12 K_B + 12\Lambda)$$

$$\begin{array}{ccc} \tilde{Y} & & J(\tilde{Y}) \\ \downarrow & & \downarrow \\ \tilde{B} & & \tilde{B} \end{array}$$

Blow back down

$$\begin{tikzcd} \bar{Y} \arrow[d] & J(\bar{Y}) \arrow[d] & J^0(Y) \arrow[l] \arrow[r] \arrow[d] & W \arrow[d] \\ \bar{B} \arrow[r, "\cong"] & \bar{B} & B' \arrow[l] \arrow[r, equal] & B' \end{tikzcd}$$

In dimension 3, the approach was worked out (Grassi, Gross, Dolgachev-Gross) (25 yrs ago)

<u>Example</u> (DRM & 6 others) 15/12, ...

$$\begin{array}{c} \hat{X} \\ \downarrow \\ \mathbb{P}^2 \\ {[s,t,u]} \end{array} \quad \text{with a point of order 3}$$

$$\frac{-y^2 z + x^3 + f x z + g z^3}{[0,1,0]}$$

$$-y^2 - a_3(s,t,u)\,xy - a_4(s,t,u)\,y + x^3$$

a_3, a_4 invariant under

$$[s,t,u] \longmapsto [\omega s, \omega^{-1} t, u]$$

a_3, a_4 functions of s^3, t^3, u^3, stu

There is a $\mathbb{Z}_3$ action on $\mathbb{P}^2$, lifts to action α on $\hat{X}$, fixes the "0-section"

$$\beta = (\alpha, \text{translation by } (0,0))$$

(using group law)

$$\begin{array}{c} \hat{X}/\beta \\ \downarrow \\ \mathbb{P}^2/\mathbb{Z}_3 \end{array}$$

there is no section, genus 1-fibered CY nonsingular

$$\pi_1(\hat{X}/\beta) = \mathbb{Z}_3$$

$$\begin{array}{c} \hat{X}/\alpha = J(\hat{X}/\beta) \\ \downarrow \\ \mathbb{P}^2_1/\mathbb{Z}_3 \end{array}$$

Singular over fixed points on $\mathbb{P}^2/\mathbb{Z}_3$.