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Algebraic Geometry Talk - Dave Morrison Jan. 29, 2016

Elliptic Calabi Yau Torus

X has dim d over \mathbb{C}
B has dim d over \mathbb{C}

E.C.Y in physics applications, F-theory

Given G.C.Y variety, there is a Weierstrass model

$$X \xrightarrow{\pi} B$$

$$y^2 = x^3 + f(x) + g$$

sections of a line bundle on the base B

$\pi^{-1}(b)$ is an elliptic curve

Theoretically

$$\begin{array}{ccc}
 \mathcal{L}_1 \otimes \mathcal{L}_2 & \subseteq & \mathcal{P}(\mathcal{O} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \\
 \downarrow & & \downarrow \\
 B & & B
 \end{array}$$

Data of Weier. model used in F-theory for compactifications

↓
10 - 2d dimensions (so don't go over $d=5$)

$S^1 \times \mathbb{R}^{8-2d, 1} \rightsquigarrow$ "M-theory compactified on S^1 "

Are there discrete gauge theories?

↳ are there multiple X 's w/ same sig?

if so, \mathbb{Z} discrete symmetry of order k

Classify all X with given f.g. in

Weierstrass model \mathbb{A}^1 where $X \rightarrow B$, $\pi^{-1}(b) \cong \mathbb{A}^1$ an elliptic

Birational geometry of X

$K_X \cong 0$; $\omega_X \cong \mathcal{O}_X$; \exists nowhere vanishing
holo (1,1)-form on X

X should be a Mori minimal model

(only mild singularities are allowed)

Any alg. curve $C \subset X$, $\deg \omega_X|_C \geq 0$.

$X \xrightarrow{\pi} B$; π is a genus one fibration if

$g(\pi^{-1}(b)) = 1$ for general b

π is an elliptic fibration if there is a rational

section: $S \subset X$ (i.e. $\deg \mathcal{O}_{\pi^{-1}(b)}(S) = 1$)
 \downarrow
 B

This means $S \rightarrow B$ is generally 1-1, onto

(2) This determines an algebraic curve of genus 1
over $k = k(B)$, if π is genus zero.

We get an actual elliptic curve k

e.g. if $k = \mathbb{Q}$, $y^2 = x^4 + 2$ (genus 1 curve)

$$V(-y^2 + x^4 + 2z^4) \subseteq \mathbb{P}^{(1,2,1)}$$

this has no \mathbb{Q} -rational point

$$\frac{a^2}{b^2} = \frac{c^4}{d^4} + 2$$

This curve has a Jacobian $J(C)$:

$$y^2 = x^3 - 8x$$

$$\frac{a^2 d^4 - b^2 c^4 - 2b^2 d^4}{b^2 d^4}$$

$$a^2 d^4 - b^2 c^4 = 2b^2 d^4$$

~~AB~~

The Jacobian has a rational point.

The group-scheme $J(C)$ acts on C :

$$J(C) \times C \rightarrow C$$

~~this has~~

$\Rightarrow C$ is a principal homog. space on
torsion over $J(C)$.

Elliptic curve $E/\mathbb{C} \rightsquigarrow$ Weierstrass model

$p \in E \rightsquigarrow \mathcal{O}_E(p)$ (line bundle, w/ dir p)

\searrow
 $\mathcal{O}_E(nP)$

by Riemann-Roch:

$$\dim \Gamma(\mathcal{O}_E(nP)) = n - |c| = n$$

$\Gamma(\mathcal{O}_E(p)) \ni 1$ (generating section of line bundle)
in $\mathbb{C}P^1$

(trivialize $\mathcal{O}_E(p)$ on $E \setminus \{p\}$)

$$\Gamma(\mathcal{O}_E(2P)) \ni 1^2, x$$

$$\Gamma(\mathcal{O}_E(3P)) \ni 1^3, x, y$$

$$\Gamma(\mathcal{O}_E(4P)) \ni 1^4, x^2, y, x^2$$

$$\Gamma(\mathcal{O}_E(5P)) \ni 1^5, x^3, x^2y, x^2, xy$$

$$\Gamma(\mathcal{O}_E(6P)) \ni 1^6, x^4, x^3y, x^2y, x^3, y^2$$

$$\omega_E \cong \mathcal{O}_E$$

$$O = -y^2 - \lambda x^3$$

built by $\mu^2 X^3 \Rightarrow \mu^2 X^3 y^2 - (\mu^2 y)^2 - \dots$

$$O = -y^2 - a_1 xy - a_3 y + x^3 + a_2 x^2 + a_4 x + a_6$$

If $\text{char } k \neq 2, 3 \Rightarrow$ complete square to get

$$O = \left(y - \frac{1}{2} a_1 x - \frac{1}{2} a_3\right)^2 + x^3 + \left(a_2 + \frac{1}{4} a_1^2\right) x^2 + \left(a_4 + \frac{1}{2} a_1 a_3\right) x + \left(a_6 + \frac{1}{4} a_3^2\right)$$

\Rightarrow complete cube ...

$$-\left(y + \frac{1}{2} a_1 x + \frac{1}{2} a_3\right)^2 + \left(x + \frac{1}{3} a_2 + \frac{1}{2} a_1\right)^3 + f_4 \left(x + \frac{1}{2} a_1\right) + g_6$$

$$O = -y^2 + x^3 + f_4 x + g_6$$

$$(x, y) \mapsto (u^2 x, u^3 y)$$

$$u^{-6} \left((-u^3 y)^2 + (u^2 x)^3 + f_4 (u^2 x) + g_6 \right) =$$

$$-y^2 + x^3 + u^{-4} f_4 x + u^{-6} g_6$$

$K \supset \mathcal{O}$ ($\mathcal{O} = \mathcal{O}_u$ for some open $u \subset B$)

$f_4, g_6 \in K \Rightarrow$ Choosing some $u \in \mathcal{O}$, we can guarantee $u^{-4} f_4, u^{-6} g_6 \in \mathcal{O}$.

\therefore WLOG, $f_4, g_6 \in \mathcal{O}$

lf $\exists v \in \mathcal{O}$ s.t. $v^4 | f_4, v^6 | g_6$

We can change coordinates $(f_4, g_6) \mapsto \left(\frac{f_4}{v^4}, \frac{g_6}{v^6}\right)$

Lemma

$\exists f_4, g_6$ s.t. $v^4 | f_4$ & $v^6 | g_6 \Rightarrow v^4, g_6$ with $v \in \mathcal{O}$

\Rightarrow (Weierstrass) minimal model

$f_4, g_6 \in \mathcal{O}_u, u \subset B$

$$V(-y^2 + x^3 + f_4 x + g_6) \subseteq \mathcal{O}_u \oplus \mathcal{O}_u$$

\downarrow
 u



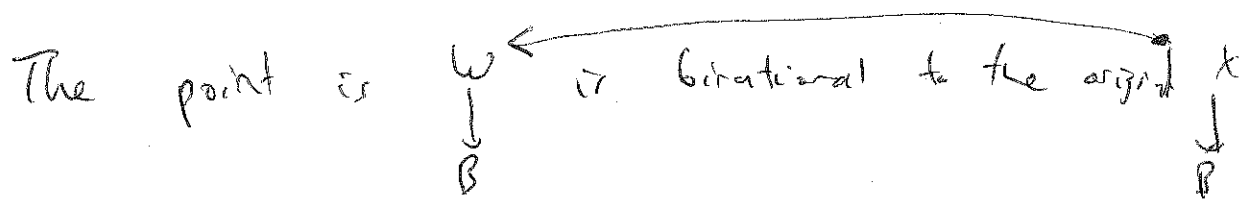
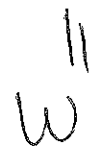
$$f_4 \in \Gamma(L^{\otimes 4})$$

$$g_6 \in \Gamma(L^{\otimes 6})$$

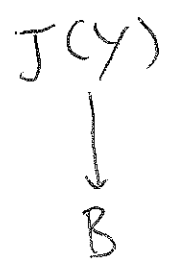
$$x \in \Gamma(L^{\otimes 2}) \quad y \in \Gamma(L^{\otimes 3})$$

Standard way of projectivizing:

$$V(-y^2 + x^3 + f_4 x z^4 + g_6 z^6) \subseteq \mathbb{P}(\mathcal{O} \oplus L^{\otimes 2} \oplus L^{\otimes 3})$$



genus 1 fibration



$C =$ genus 1 curve over k

$$\Rightarrow J(C) = \{ \text{line bundles of degree 0 on } C \}$$

$=$ Picard scheme of C

$[O_C] \in J(C)$; this contains the data of an elliptic curve

$$\begin{array}{ccc}
 J(C) \times C & \longrightarrow & C \\
 \downarrow & \downarrow & \\
 L & \downarrow & p \longmapsto \mathcal{L} \otimes \mathcal{O}(p) \cong \mathcal{O}(q)
 \end{array}$$

where $J(C) \otimes \mathcal{O}(p) \cong \mathcal{O}(q)$

$[L_1], [L_2] \in J(C) \Rightarrow [L_1 \otimes L_2]$ is product

$$\begin{array}{c}
 Y \\
 \downarrow \\
 B
 \end{array}$$
 genus 1 fibration

Modify Y birationally, as well as B , into a form in which the Jacobian of Y can be easily defined.

Also in this form, we get a formula:

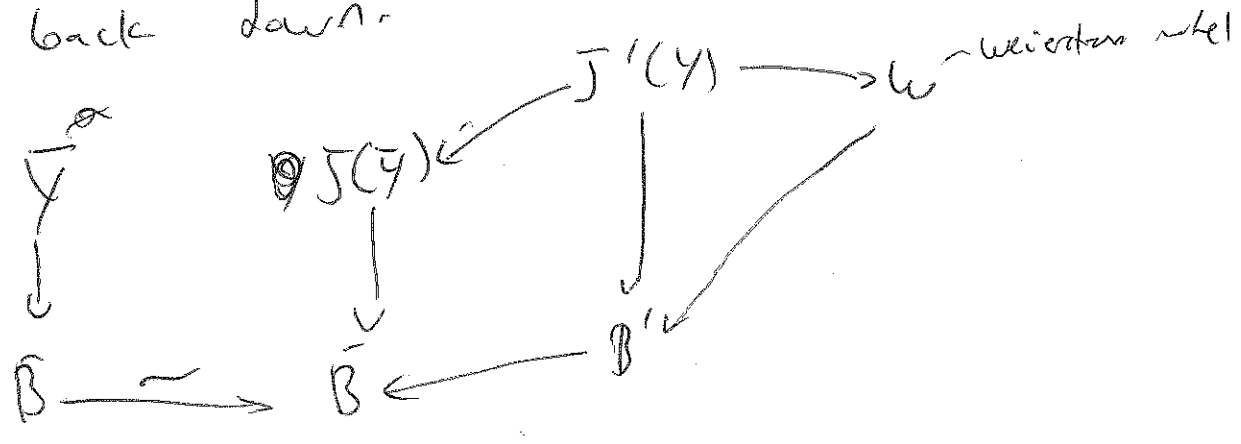
$$\omega_Y = \pi^*(\omega_B(\Delta))$$

\mathcal{L} has \mathbb{Q} -coefficients

$$K_Y = \pi^*(K_B + \Delta) ; 12K_Y = \pi^*(12K_B + 12\Delta)$$

After getting \tilde{Y} and $J(Y)$ birationally,
 $\tilde{Y} \downarrow B$ and $J(Y) \downarrow B$

(i.e. blow up base and fiber), we want to
blow back down.



$Y =$ minimal model ; \tilde{Y} blow up

Given $J'(Y)$ model, what are all Y 's that can happen?

If Y has $\dim. \leq 4$, done. $\dim Y \geq 5$.

by Bramm, Gross, Dolgachev-Gross

Example (PRM + 6 others)

~~PRM~~

$J'(Y)$ not a minimal model
 is the surprise

Start w/ X



with a point of order 3

$\mathbb{P}^2_{[s,t,u]}$

~~$x^3 + y^3 + z^3 + g_2 x^2 y + g_3 x y^2 - y^2 z + x^3 + f_1 z + g_1 x z$~~

$[0, 1, 0]$

$-y^2 - g_2(s, t, y)xy - a_1(s, t, u)y + x^3$

Choose a_2, a_1 to be invariant w.r.

$[s, t, u] \mapsto [w s, w^{-1} t, u]$

i.e. a_2, a_1 functions of s^3, t^3, u^3, st^4

$\mathbb{Z}/3$ action α on \mathbb{P}^2 , lifts to action on X fixes the 0-section

$\mathbb{Z}/3$ action β ; do d, then translation $g_1(0,0)$

~~PRM~~

⑥

X/β has no sections; genus 1. fibered C

$$\downarrow \\ \mathbb{P}^2/\mathcal{O}_3$$

$$X/\alpha = J(X/\beta) \quad \text{with } \mathcal{O}_3$$

$$\downarrow \\ \mathbb{P}^2/\mathcal{O}_3$$

$$\pi_1(X/\beta) = \mathcal{O}_3$$

Singular over fixed points on $\mathbb{P}^2/\mathcal{O}_3$

$$\text{Blow up } \mathbb{P}^2/\mathcal{O}_3 \rightsquigarrow \widetilde{\mathbb{P}^2/\mathcal{O}_3} \rightsquigarrow J(X/\beta)$$

$$[0, 1, 0] = "0"$$

$[0, 0, 1] = \text{point of order 3}$

