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Algebraic Geometry Talk - Dave Morrison Jan. 29, 2016

Elliptic Calabi Yau Torses

X has dim $d+1$ over $\mathbb{C}$
B has dim d over $\mathbb{C}$

E.C.Y $\rightsquigarrow$ physics applications, F-theory

Given E.C.Y variety, there is a Weierstrass model

$X \xrightarrow{\pi} B$

$$y^2 = x^3 + fx + g$$

section of a line bundle on the base B

$\pi^{-1}(b)$ is an elliptic curve

$$\begin{array}{ccc} \mathcal{L}_1 \oplus \mathcal{L} & \subseteq & \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \\ \downarrow & & \downarrow \\ B & & B \end{array}$$

Data of Weier. model used in F-theory for compactification

$\rightsquigarrow$ $10 - 2d$ dimensions (so don't go over $d=5$)

$S^1 \times \mathbb{R}^{8-2d,1}$ $\rightsquigarrow$ "M-theory compactified on X"

Are there discrete gauge theories?

$\hookrightarrow$ are there multiple X's w/ same f,g?

If so, $\exists$ discrete symmetry of order k

Classify all $\cancel{X}$ with given f, g in Weierstrass model $\cancel{X}$ where $X \to B$, $\pi^{-1}(b) \to$ an ellipt.

Birational geometry of $\cancel{X}$

$K_X \equiv 0$; $\omega_X \cong \mathcal{O}_X$; $\exists$ nowhere vanishing holo $(d+1)$-form on X

X should be a Mori minimal model

(only mild singularities are allowed)

Any alg. curve $C \subset X$, $\deg \omega_X|_C \geq 0$.

$X \xrightarrow{\pi} B$; π is a <u>genus one fibration</u> if

$g(\pi^{-1}(b)) = 1$ for general b

π is an <u>elliptic fibration</u> if there is a rational section :

$$\begin{array}{ccc} S \subset & X & \\ & \downarrow & \\ & B & \end{array}$$

(i.e. $\deg \mathcal{O}_{\pi^{-1}(b)}(S) = 1$)

This means $S \to B$ is generally 1-1, onto

(2) This determines an algebraic curve of genus 1 over $K = K(B)$, if π is genus one.

We get an actual elliptic curve K

e.g. If $K = \mathbb{Q}$, $y^2 = x^4 + 2$ (genus 1 curve)

$$V(-y^2 + x^4 + 2z^4) \subseteq \mathbb{P}^{(1,2,1)}$$

this has no $\mathbb{Q}$-rational point

$\frac{a^2}{b^2} = \frac{c^4}{d^4} + 2$

$\frac{a^2 d^4 - b^2 c^4 - 2b^2 d^4}{b^2 d^4}$

$a^2 d^4 - b^2 c^4 = 2b^2 d^4$

This curve has a Jacobian $J(C)$:

$$y^2 = x^3 - 8x$$

The Jacobian has rational points.

The group-scheme $J(C)$ acts on C:

$$J(C) \times C \longrightarrow C$$

~~[illegible]~~

$\Rightarrow$ C is a principal homog. space or torsor over $J(C)$.

Elliptic curve/k $\rightsquigarrow$ Weierstrass model

$p \in E \rightsquigarrow \mathcal{O}_E(p)$ (line bundle, w/ div p)

$\searrow \mathcal{O}_E(nP)$

By Riemann Roch:

$$\dim \Gamma(\mathcal{O}_E(nP)) = n - 1 + 1 = n$$

$\Gamma(\mathcal{O}_E(p)) \ni \underline{1}$ (generating section of line bundle on nbhd)

(trivializes $\mathcal{O}_E(p)$ on $E \setminus \{p\}$)

$\Gamma(\mathcal{O}_E(2P)) \ni 1^2, x$

$\Gamma(\mathcal{O}_E(3p)) \ni 1^3, 1 \cdot x, y$

$\Gamma(\mathcal{O}_E(4p)) \ni 1^4, 1^2 x, 1y, x^2$

$\Gamma(\mathcal{O}_E(5p)) \ni 1^5, 1^3 x, 1^2 y, 1x^2, xy$

$\Gamma(\mathcal{O}_E(6p)) \ni 1^6, 1^4 x, 1^3 y, 1^2 x^2, 1xy, x^3, y^2$

$\omega_E \simeq \mathcal{O}_E$ $\qquad 0 = -y^2 - \qquad x^3$

$\lambda \qquad \mu$

(mult by $\mu^2 \lambda^3 \Rightarrow \mu^2 \lambda^3 y^2 \cdots$)

$(\mu\lambda^2 y)^2 \cdots$

$$0 = -y^2 - a_1 xy - a_3 y + x^3 + a_2 x^2 + a_4 x + a_6$$

If char $k \neq 2, 3 \Rightarrow$ complete square to get

$$0 = (y - \tfrac{1}{2} a_1 x - \tfrac{1}{2} a_3)^2 + x^3 + (a_2 + \tfrac{1}{4} a_1^2) x^2 +$$

$$(a_4 + \tfrac{1}{2} a_1 a_3) x + (a_6 + \tfrac{1}{4} a_3^2)$$

$\Rightarrow$ complete cube ...

$$-(y + \tfrac{1}{2} a_1 x + \tfrac{1}{2} a_3)^2 + (x + \tfrac{1}{3} a_2 + \tfrac{1}{2} a_1)^3 + f_4(\underbrace{\quad}_{x + \frac{1}{2} a_2 + \frac{1}{12} a_1^2}) + g_6$$

$$0 = -y^2 + x^3 + f_4 x + g_6$$

$(x, y) \mapsto (u^2 x, u^3 y)$

$u^{-6} \left((-u^3 y)^2 + (u^2 x)^3 + f_4 (u^2 x) + g_6 \right) =$

$-y^2 + x^3 + u^{-4} f_4 x + u^{-6} g_6$

$K \supset \mathcal{O}$ $\quad (\mathcal{O} = \mathcal{O}_U$ for some open $U \subset B)$

$f_4, g_6 \in K \Rightarrow$ Choosing some $u \in \mathcal{O}$, we can guarantee $u^{-4} f_4, u^{-6} g_6 \in \mathcal{O}$

$\therefore$ WLOG, $f_4, g_6 \in \mathcal{O}$

If $\exists\, v \in \mathcal{O}$ s.t. $v^4 \mid f_4$, $v^6 \mid g_6$

we can change coordinates $(f_4, g_6) \mapsto \left(\frac{f_4}{v^4}, \frac{g_6}{v^6}\right)$

Lemma

$\nexists$ f_4, g_6 s.t. $v^4 \mid f_4$ & $v^6 \mid g_6 \Rightarrow v^4, g^6$ units in $\mathcal{O}$

$\Rightarrow$ (Weierstrass) minimal model

$f_4, g_6 \in \mathcal{O}_U$, $U \subset B$

$$\begin{array}{c} V(-y^2 + x^3 + f_4 x + g_6) \subseteq \mathcal{O}_U \oplus \mathcal{O}_U \\ \downarrow \\ U \end{array}$$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ B \end{array} \qquad \begin{array}{l} f_4 \in \Gamma(\mathcal{L}^{\otimes 4}) \\ g_6 \in \Gamma(\mathcal{L}^{\otimes 6}) \\ x \in \Gamma(\mathcal{L}^{\otimes 2}) \quad y \in \Gamma(\mathcal{L}^{\otimes 3}) \end{array}$$

Standard way of projectivizing:

$$\begin{array}{ccc} V(-y^2 + x^3 + f_4 x z^4 + g_6 z^6) & \subseteq & \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \\ \| & & \downarrow \\ W & & B \end{array}$$

The point is $\begin{array}{c} W \\ \downarrow \\ B \end{array}$ is birational to the original $\begin{array}{c} X \\ \downarrow \\ \mathbb{P} \end{array}$

$\begin{array}{c} Y \\ \downarrow \\ B \end{array}$ genus 1 fibration $\qquad \begin{array}{c} J(Y) \\ \downarrow \\ B \end{array}$

$C =$ genus 1 curve over k

$\Rightarrow J(C) = \{$ line bundles of degree 0 on $C \}$

$=$ Picard scheme of C

$[\mathcal{O}_C] \in J(C)$; this contains the data of an elliptic curve

$$\begin{array}{ccc} J(C) \times C & \longrightarrow & C \\ \rotatebox{90}{$\in$}\ \ \ \rotatebox{90}{$\in$} & & \\ \mathcal{L} \quad\ p & \longmapsto & \cancel{\mathcal{L} \otimes \mathcal{O}(p) \cong \mathcal{O}(q)}\ q \end{array}$$

where $J(C) \otimes \mathcal{O}(p) \cong \mathcal{O}(q)$

$[\mathcal{L}_1], [\mathcal{L}_2] \in J(C) \Rightarrow [\mathcal{L}_1 \otimes \mathcal{L}_2]$ is product

$$\begin{array}{c} Y \\ \downarrow \\ B \end{array} \quad \text{genus 1 fibration}$$

Modify Y birationally, as well as B, into a form in which the Jacobian of Y can be easily defined.

Also in this form, we get a formula:

$$\omega_Y = \pi^*(\omega_B(\underbrace{\Lambda}_{\text{has } \mathbb{Q}\text{-coefficients}}))$$

$$K_Y = \pi^*(K_B + \Lambda) \ ; \ 12K_Y = \pi^*(12K_B + 12\Lambda)$$

After getting $\begin{array}{c}\tilde{Y}\\ \downarrow \\ \tilde{B}\end{array}$ and $\begin{array}{c}J(\tilde{Y})\\ \downarrow \\ \tilde{B}\end{array}$ birationally,

(i.e. blow up base and fiber), we want to

blow back down.

$$
\begin{tikzcd}
 & & J'(Y) \arrow[dl] \arrow[dd] \arrow[r] & W \arrow[ddl] \\
\bar{Y} \arrow[d] & J(\bar{Y}) \arrow[d] & & \\
\tilde{B} \arrow[r, "\sim"] & \bar{B} & B' \arrow[l] &
\end{tikzcd}
$$

W – Weierstrass model

$\bar{Y}$ = minimal model ; $\tilde{Y}$ blow up

Given $J'(Y)$ model, what are all $\tilde{Y}$'s that can happen?

If Y has dim. ≤ 4, done. dim 4 open.

by Grassi, Gross, Dolgachev–Gross

Example (DRM + 6 others)

$\boxed{J'(Y) \text{ not a minimal model is the surprise}}$

Start w/

$$\hat{X} \to \mathbb{P}^2_{[s,t,u]}$$

with a point of order 3

$\cancel{y^2 + x^3 + f x z^4 + g z^6}$ $-y^2 z + x^3 + f x z + g x \; z$

$[0,1,0]$

$-y^2 - a_3(s,t,u)\,xy - a_q(s,t,u)\,y + x^3$

Choose a_3, a_q to be invariant under

$$[s,t,u] \mapsto [\omega s, \omega^{-1} t, u]$$

i.e. a_3, a_1 functions of s^3, t^3, u^3, stu

$\mathbb{Z}/3$ action α on $\mathbb{P}^2$, lifts to action on $\hat{X}$ fixes the 0-section

$\mathbb{Z}/3$ action β: do α, then translation by $(0,0)$

$$\begin{array}{c} \tilde{X}/\beta \\ \downarrow \\ \mathbb{P}^2/\mathbb{Z}_3 \end{array}$$ has no section; genus 1 fibered CY

$$\begin{array}{c} \tilde{X}/\alpha = J(\tilde{X}/\beta) \\ \downarrow \\ \mathbb{P}^2_1/\mathbb{Z}_3 \end{array}$$

$\pi_1(\tilde{X}/\beta) = \mathbb{Z}_3$

Singular over fixed points on $\mathbb{P}^2/\mathbb{Z}_3$

Blow up $\mathbb{P}^2/\mathbb{Z}_3 \rightsquigarrow \widetilde{\mathbb{P}^2/\mathbb{Z}_3} \rightsquigarrow J(\tilde{X}/\beta)$

$[0,1,0] = $ "∞"

$[0,0,1] =$ point of order 3