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Knot contact homology and the augmentation polynomial

Outline

- KCH
- aug poly
- stable A-polynomial
- HOMFLY / physics

Setup $K \subset M^n$

K: submanifold; M^n: smooth

T^*M^{2n} is naturally a symplectic manifold $\omega \in \Omega^2(T^*M)$, $\omega = \sum dq_i \wedge dp_i$

ST^*M^{2n} is naturally a contact manifold $\alpha \in \Omega^1(ST^*M)$, $\alpha = \sum p_i \, dq_i$

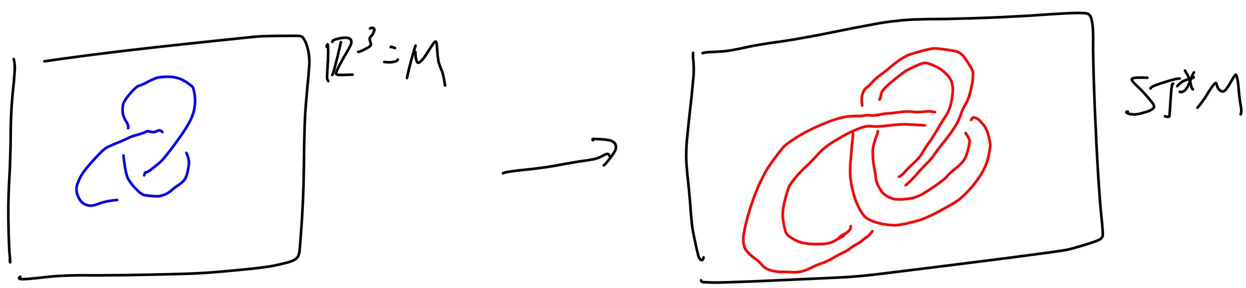
Def $L_K = \{(q,p) \in T^*M \mid q \in K, \langle p, v\rangle = 0 \ \forall v \in T_q K\} \subseteq T^*M$

is the conormal bundle to K

$\Lambda_K = L_K \cap ST^*M \subseteq ST^*M$, the unit conormal bundle

Observation:

- L_K is Lagrangian in T^*M: $\omega|_{L_K} \equiv 0$
- Λ_K is Legendrian in ST^*M: $\alpha|_{\Lambda_K} \equiv 0$



$K_1 \simeq K_2$ isotopic $\Longrightarrow$ $\Lambda_{K_1} \simeq \Lambda_{K_2}$ Legendrian isotopic

$$K \rightsquigarrow \Lambda_K \rightsquigarrow LCH(\Lambda_K)$$

Legendrian contact homology

Specialize: K = knot in $M = \mathbb{R}^3$, for the rest of the talk.

Def The knot contact homology of K is

$$HC_*(K) = LCH(\Lambda_K)$$

an invariant of smooth knots K.

$$HC_*(K) = H_*(\underbrace{\mathcal{A}_K, \partial_K}_{\text{differential graded algebra associated to } K})$$

$\mathcal{A}$ = tensor algebra over a coefficient ring

$$R := \mathbb{Z}[\underbrace{H_2(ST^*\mathbb{R}^3, \Lambda_K)}_{H_2(S^2) \oplus H_1(T^2)}]$$

$$= \mathbb{Z}[Q^{\pm 1}, \lambda^{\pm}, \mu^{\pm}]$$

λ, μ = longitude, meridian

generated by finitely many "Reeb chords" of Λ_K.

Differential ∂_K counts holomorphic disks in $\mathbb{R} \times ST^*\mathbb{R}^3$

with boundary on $\mathbb{R} \times \Lambda_K$.

$\partial_K^2 = 0$, $|\partial_K| = -1$.

Example Unknot U

deg 1 deg 2

$$\mathcal{A}_U = R\langle a_1, a_2 \rangle$$

$$\partial_U(a_1) = Q - \lambda - \mu + \lambda\mu$$

$$\partial_U(a_2) = 0$$

Properties of $(\mathcal{A}_K, \partial_K)$

1. Knot invariant up to equivalence
2. $\exists$ combinatorial form for $(\mathcal{A}_K, \partial_K)$
   (Ekholm, Etnyre, N, Sullivan)
3. contains $\Delta_K(t)$

Augmentations

Def An augmentation of a DGA $(\mathcal{A}, \partial)$ over R

is a graded ring homomorphism

$$\epsilon : \mathcal{A} \to \mathbb{C} \quad \text{has degree 0}$$

such that $\epsilon \circ \partial = 0$.

Note Any augmentation ϵ restricts to $\epsilon|_R : R \to \mathbb{C}$

i.e. $\epsilon|_R \in \mathrm{Hom}(R, \mathbb{C}) \cong (\mathbb{C}^*)^3$

$\curvearrowleft$ $\mathbb{C} \setminus 0$

Def The augmentation variety of K is

$$V_K = \text{highest dimensional part of closure of}$$
$$\{\epsilon|_R : \epsilon = \text{aug of } (\mathcal{A}_K, \partial_K)\} \subset (\mathbb{C}^*)^3$$

Ex $V_U = \{Q - \lambda - \mu + \lambda\mu = 0\}$

Observation In all known examples, V_K is a codim 1 subvariety of $(\mathbb{C}^*)^3$

Def The augmentation polynomial $\mathrm{Aug}_K(Q, \lambda, \mu)$ is the minimal polynomial with $V_K = \{\mathrm{Aug}_K = 0\}$.

Relation to A-polynomial

$A_K(\lambda, \mu)$: $\quad \rho : \underbrace{\pi_1(S^3 \setminus K)}_{\substack{m,\ \ell \text{ meridian, longitude} \\ \text{(which commute)}}} \to SL_2\mathbb{C}$

$$\rho(m) = \begin{bmatrix} \mu & * \\ 0 & \mu^{-1} \end{bmatrix}, \quad \rho(\ell) = \begin{bmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{bmatrix}$$

Def highest dimensional part of the closure of

$$\{(\lambda, \mu) \mid \rho = SL_2\mathbb{C} \text{ rep}\} \subset (\mathbb{C}^*)^2$$

is the vanishing set of the A-polynomial $A_K(\lambda, \mu)$

Generalize

Def $\tilde{\rho} : \pi_1(S^3 \setminus K) \longrightarrow GL_n\mathbb{C}$, $n \geq 2$

with

$$\tilde{\rho}(m) = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

identity matrix

$$\tilde{\rho}(l) = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}$$

is called a KCH-representation.

(can renormalize an $SL_2\mathbb{C}$-rep to get a KCH-rep with $n=2$)

Def highest-dim'l piece of closure of $\{(\lambda,\mu) \mid \tilde{\rho} = \text{KCH-rep for any } n\} \subseteq (\mathbb{C}^*)^2$

is the vanishing set of the stable A-polynomial

$\tilde{A}_K(\lambda,\mu)$

Prop

1. $SL_2\mathbb{C}$-rep $\Rightarrow$ KCH rep.

   $\Rightarrow \quad A_K(\lambda,\mu^{1/2}) \mid \tilde{A}_K(\lambda,\mu)$

2. KCH-rep $(\lambda,\mu) \Rightarrow$ augmentation ϵ of $(\mathcal{A}_K, \partial_K)$

   with $\epsilon(\lambda)=\lambda$, $\epsilon(\mu)=\mu$, $\epsilon(Q)=1$.

   "$\Rightarrow$" $\tilde{A}_K(\lambda,\mu) \mid \mathrm{Aug}_K(\lambda,\mu,1)$

Cor Aug_K detects the unknot

Prop (Cornwell)
1) For fixed K, the n in KCH-rep is bounded above.
2) In #2 of previous proposition, $\Longleftarrow$ is also true

Cor Up to repeated factors, $Aug_K(\lambda, \mu, 1) = \tilde{A}_K(\lambda, \mu)$

Relation to HOMFLY

K = knot

$P_{K,n}(a,q)$ = colored HOMFLY poly of K, colored by the n^{th} symm. power of fundamental representation

Prop (?) (Garoufalidis) $\{P_{K,n}(a,q)\}_{n=1}^{\infty}$ is q-holonomic: i.e., satisfy a recursion

Define operators L, M by

$$L(P_{K,n}(a,q)) = P_{K,n+1}(a,q)$$

$$M(P_{K,n}(a,q)) = q^n P_{K,n}(a,q)$$

$\exists$ (minimal nonzero) recurrence of the form

$$\hat{A}_K(a,q,M,L)\, P_{K,n}(a,q) = 0$$

poly in commuting variables a, q
non-commuting variables M, L $(ML = qLM)$

Conj (Aganagic, Ekholm, N, Vafa)

$$\hat{A}_K|_{q=1}(a, M, L) = Aug_K(\lambda, \mu, Q)$$

$\Rightarrow$ "the augmentation polynomial is the classical limit of the recurrence relation for colored HOMFLY."

cf. AJ conjecture:

"A-poly is classical limit of recurrence for colored Jones"