F-theory and Elliptic Fibrations

Dave Morrison

IIB String Theory:

complex valued scalar field τ

$SL(2,\mathbb{Z})$ $\quad \tau \mapsto \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

Duality: M-theory on T^2 w ~~metric~~ metric

conformal class $\longleftrightarrow$ ~~Cx~~ structure + area

$\longleftrightarrow$ labeled by

so M-theory on $T^2 \cong$ IIB on S^1

area$(T^2) \sim \ell(S^1)^{-4/3}$. $\quad$ CX structure $\tau \longleftrightarrow$ scalar field τ

CX $\longleftrightarrow$ τ mod $SL(2,\mathbb{Z})$

$$T^2 \cong \mathbb{C} \big/ \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{C}/\Lambda \qquad \tau \equiv \frac{\omega_2}{\omega_1}$$

change of basis on $\Lambda \longleftrightarrow SL(2,\mathbb{Z})$ trans.

$T^2 \mapsto 0 \iff \ell(S^1) \to \infty$ get IIB.

F-theory $\quad$ families of T^2's

$\updownarrow$

IIB w. variable τ, exploit $SL(2,\mathbb{Z})$

D7-brane: real codim 2 defect

$\tau \to \tau + 1$

IIB on S (or on $S \setminus \Delta$)

(Calabi-Yau for SUSY)

alg. geom.
$$\left\{\begin{array}{ccc} X & \supset & X \setminus \pi^{-1}(\Delta) \\ \downarrow\pi & & \downarrow \\ S & \supset & S\setminus\Delta \end{array}\right\} T^2 \text{ fiber bundle}$$

* away from Δ, T^2 fibers
* on Δ fiber is singular

<u>Weierstrass</u> $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

non-constant meromorphic function

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{(0,0)\neq(m,n)\in\Lambda}\left(\frac{1}{(z-n\omega_1-m\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2}\right)$$

$$\wp(z;\omega_1,\omega_2) = z^{-2} + 3G_4(\omega_1,\omega_2)z^2 + 5G_6(\omega_1,\omega_2)z^4 + \mathcal{O}(z^6)$$

$$G_{2k}(\omega_1,\omega_2) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1+n\omega_2)^{2k}}$$

$$\left(\tfrac{1}{2}\wp'\right)^2 - \wp^3 - 15G_4\wp - 35G_6 = \text{regular at } \{z=0\}$$

$\wp \mapsto x$, $\wp' \mapsto y$ $\qquad \mathcal{G}_1^2 \qquad \equiv 0$ (vanishes at $\{z=0\}$)

$y^2 = x^3 + fx + g$ describes $(\mathbb{C}\setminus\Lambda)/\Lambda$ — f and g are explicit functions of ω_1, ω_2

compactification $\left(y^2 = x^3 + fxz^4 + gz^6\right) \subseteq \mathbb{CP}^2$

$x \mapsto \lambda^2 x \quad y \mapsto \lambda^3 y \quad f \mapsto \lambda^4 f \quad g \mapsto \lambda^6 g$

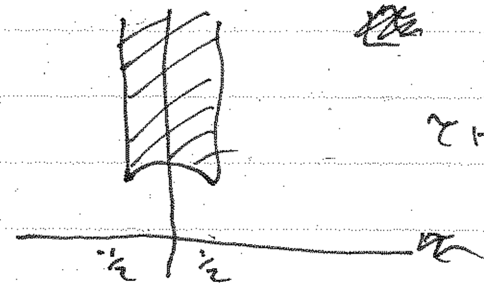
preserves equation

$\{y^2 z = x^3 + f x z^2 + g z^3\} \subseteq \mathbb{CP}^2$

$$\omega_1 = \int_{\gamma_1} \frac{dx}{y} \qquad \omega_2 = \int_{\gamma_2} \frac{dx}{y}$$

* let f, g be "functions" on S

$$\begin{array}{c} X \\ \pi \downarrow \\ S \end{array}$$



$\tau \mapsto \tau + 1 \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\operatorname{Im} \tau \to \infty \quad \cancel{q} \quad q = e^{2\pi i \tau} \quad q \to 0$

$\mathbb{C}/\Lambda$ is a curve (Riemann surface) over $\mathbb{C}$ of genus 1.

$$\begin{array}{c} X \\ \pi \downarrow \\ S \end{array}$$

S: alg. variety (defined by poly in projective space)

X: alg. variety

$\exists\, \pi \Rightarrow$ describe X in terms of S "local coords"

$X \Leftrightarrow$ curve over the field of rational functions on S.

genus 1 curves over field K. $\operatorname{char} K = 0$.

Tools from algebraic geometry

Riemann-Roch theorem

curve C; $D = \sum n_p P$ point P.

$H^0(\mathcal{O}(D)) = \{ D' = \sum n_p' P \mid D' - D = \text{zeros - poles of meromorphic func. and } n_p' \geq 0 \}$

THM

$\dim(H^0(\mathcal{O}(D))) - \dim H^0(\mathcal{O}(\text{zeros of mero 1-form}) - 1)$

$= \deg D - g + 1.$

Assume C has a ~~point~~ point $P \in C$

observe $y^2 z = x^3 + f x z^2 + g z^3 \subseteq \mathbb{CP}^2$

ψ (in our example)

$[0,1,0]$

$D = P \quad D = 2P \quad D = 3P \quad D = 4P$

meromorphic 1-form on C has equal numbers of zeros and poles

* need $\dim H^0(\mathcal{O}(D)) = \deg D$

$H^0(\mathcal{O}(P)) \ni w$

$H^0(\mathcal{O}(2P)) \ni w^2, x$

$H^0(\mathcal{O}(3P)) \ni w^3, wx, y$

$H^0(\mathcal{O}(4P)) \ni w^4, w^2x, wy, x^2$

$H^0(\mathcal{O}(5P)) \ni w^5, w^3x, w^2y, wx^2, xy$ ← 7 gens; 6D space $\Rightarrow$ relation

$H^0(\mathcal{O}(6P)) \ni w^6, w^4x, w^3y, w^2x^2, wxy, x^3, y^2$

complete the square

$$\underbrace{y^2 + a_1 xyw + a_3 yw^3}_{} = x^3 + a_2 x^2 w^2 + a_4 xw^4 + a_6 w^6$$

$\subseteq \mathbb{P}^{(1,2,3)}$ (z, x, y) $\Rightarrow$ elliptic curve

$$\tilde{y}^2 = \underbrace{x^3 + \tilde{a}_2 x^2 w^2 + \hat{a}_4 xw^4 + \tilde{a}_6 w^6}_{}$$

complete cube to get rid of x^2

$\Rightarrow \; = \tilde{x}^3 + f\tilde{x} + g$

THE POINT: f and g are "sections of line bundles" on S.

$\hookrightarrow$ they are not constants on the base. (previously had constants)

- Where are the defects?

$y^2 = x^3 + fx + g$

$\hookrightarrow$ singular when ~~$2y=A$~~ $2y=0$ $\quad 3x^2+f=0$

$\Rightarrow x^3 + fx + g = 0$
$3x^2 + f = 0$ $\mapsto$ use Gröbner method.

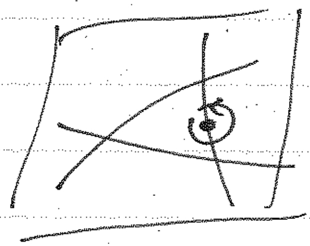
figure out what to multiply eq'ns by to subtract off equations and eliminate variables

$\underline{\Delta} = 4f^3 + 27g^2$

$$\begin{array}{c} X \\ \downarrow \pi \\ S \end{array}$$

f, g locally defined on S

* zeros of Δ = defects



* defects are where there are monodromies

THE ANSWER v1. (Kodaira / Neron)

order of vanishing

monod.	codim 2 sing	brane	Kodaira notation	ord f	ord g	ord Δ
	$\times$	$\times$	I_0	≥ 0	≥ 0	0
$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	A_{n-1}	$n D_7$	I_n	0	0	n
—	—	exotic	II	≥ 1	1	2
	A_1		III	1	≥ 2	3
	A_2		IV	≥ 2	2	4
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	D_4	$O_7 + 4D_7$	I_0^*	≥ 2	≥ 3	6
$\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$	D_{4+n}	$O_7 + (4+n) D_7$	I_n^*	2	3	$n+6$
?	E_6	exotic	IV^*	≥ 3	4	8
	E_7		III^*	3	≥ 5	9
	E_8		II^*	≥ 4	5	10
	non-minimal			≥ 4	≥ 6	≥ 12