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Non abelian GLSM
and non-complete-intersection GLSM

Gauged linear sigma model (Witten '93)

~~2D superconformal~~

2D supersymmetric field theory $\xrightarrow{IR}$ SCFT

G = compact group

$\rho : G \to GL(V)$ in practice: write a basis of V

W = superpotential = G-invt poly on V (coefs are params)

GIT G on V

$V /\!/ G$

$$
\begin{tikzcd}
\mu : V \arrow[r] \arrow[dr] & \mathrm{Lie}(G)^* \arrow[d] \\
& \mathrm{Lie}(Z(G))^*
\end{tikzcd}
$$

need to choose a value for μ

D-terms $\Leftrightarrow$ choice of value for moment map

complexified with θ-angles

Physics

what are all supersymmetric vacua?

For certain values of parameters, if $G=$ abelian get CY variety

Example 1

$$V = \mathbb{C}^6 \quad \text{basis} \quad P, \Phi_1, \dots, \Phi_5$$

$$G = U(1) \quad \text{action} \quad -5, 1, 1, \dots, 1$$

$$W = P F_5(\Phi)$$

degree 5 homogeneous
invt under G

$$\mu(P, \vec{\Phi}) = \frac{1}{2}\left(-|P|^2 + |\Phi_1|^2 + \dots + |\Phi_5|^2\right)$$

$r =$ moment map value

$r > 0 \Rightarrow \vec{\Phi} \neq \vec{0}$

$V /\!\!/ G := \mu^{-1}(r)/G$ symplectic reduction

$$\left\{ (P, \vec{\Phi}) : \frac{\partial W}{\partial P} = \frac{\partial W}{\partial \Phi} = 0 \right\}$$

$$\mathbb{P}^4 = \{|\Phi_1|^2 + \dots + |\Phi_5|^2 = c\}/U(1)$$

$p =$ coordinate in line bundle $\mathcal{O}_{\mathbb{P}^4}(-5)$

$$\frac{\partial W}{\partial p} = F(\Phi)$$

$$\frac{\partial W}{\partial \Phi_j} = p \frac{\partial F}{\partial \Phi_j}$$

F generic $\Rightarrow \dfrac{\partial F}{\partial \Phi_j} \quad \forall j$

$$\Updownarrow$$

$$\vec{\Phi} = 0$$

$\Rightarrow p = 0, \ F = 0$

$\Rightarrow$ 0 loci

$\Rightarrow$ set of vacua $\{F = 0\} \subseteq \mathbb{P}^4$
quintic CY

<u>$r < 0$</u> : $p \neq 0$, using $U(1)$ action $p = |p|$

residual $\mathbb{Z}_5$ action

$\mathbb{C}^5_{(\Phi)} / \mathbb{Z}_5$

related by blow-up

$\Rightarrow \vec{\Phi} = 0$

$W = F(\Phi)$ LG thy
LG orbifold

General version $\quad G =$ compact abelian

$$= U(1) \times \Gamma$$

$$W = P_1 F_1(\Phi) + \dots + P_k F_k(\Phi)$$

$$\begin{array}{c} U(1) \\ \vdots \\ U(1) \end{array} \begin{pmatrix} P_1 \dots & P_k & \Phi_1 \dots & \Phi_\ell \\ & & & \\ & & & \end{pmatrix} \qquad \begin{array}{c} \sum q = 0 \\ 0 \\ 0 \end{array}$$

$\oplus \mathcal{L}$

$$\left\{ \begin{array}{l} F_1 = F_2 = \dots = F_k = 0 \\ P_1 = \dots = P_k = 0 \end{array} \right\} \subset V /\!/ G \qquad \text{CICY}$$

quintic $\subseteq \mathbb{P}^4$

$[2] \cap [4] \subseteq \mathbb{P}^5$

$[3] \cap [3]$

$[2] \cap [2] \cap [3] \subseteq \mathbb{P}^6$

$[2] \cap [2] \cap [2] \cap [2] \subseteq \mathbb{P}^7$

hypersurface $\subseteq \mathbb{P}^N$

non-singular $\Rightarrow$ defined by 1 equation codim $2 \subseteq \mathbb{P}^N$ typical (Serre)

$$Y = X_1 \cap X_2$$

$\uparrow d_1$ F_1 $\quad$ $\uparrow d_2$ F_2 $\quad$ equations

Resolution of structure sheaf

$$\mathcal{O}_{\mathbb{P}}(-d_1 - d_2) \to \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

$$\begin{bmatrix} -F_2 & F_1 \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

1×2 matrix

maximal minors

rank 1

<u>Example of codim 3</u>

$$\mathcal{O}(-d_1-d_2) \to \mathcal{O}(-d_1-d_3) \oplus \mathcal{O}(-d_2-d_3) \to \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \mathcal{O}(-d_3) \to \mathcal{O}_Y \to 0$$

$$\begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \qquad \begin{bmatrix} 0 & -F_3 & -F_2 \\ -F_3 & 0 & F_1 \\ F_2 & -F_1 & 0 \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

<u>Buchsbaum - Eisenbud</u>

Assume hypotheses, codim 3 then

$$\mathcal{O}(-t-2s) \xrightarrow{g^\vee(-t-2s)} \mathcal{E}^\vee(-t-s) \xrightarrow{f} \mathcal{E}(-s) \xrightarrow{g} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

$g =$ ~~Pff~~ Pfaffian of $2p \times 2p$ matrix g^{th} row deleted

$N =$ skew-sym

rk N even

$\Rightarrow \det N = Pf(N)^2$

Koszul complex has generalization

Ex

Choose 7 skew-sym 7×7 matrices

$$A_1, \ldots \quad A_7 \qquad A_\alpha$$

$$A_\alpha^{jk} = -A_\alpha^{kj}$$

$A_1, \ldots A_7$ "generic"

$p_1 A_1 + \ldots + p_7 A_7 = 7 \times 7$ matrix polynomial entries (in $\vec{p}$)

$$\vec{p} \in \mathbb{C}^7$$

$$\operatorname{rank}(p^\alpha A_\alpha) \leq 6$$

$*$ Assume $\operatorname{rank}(p^\alpha A_\alpha) < 4 \Rightarrow \vec{p} = 0$

$$\{ \vec{p} \in \mathbb{P}^6 \mid \operatorname{rank}(p^\alpha A_\alpha) = 4 \}$$

$$\{ \vec{p} \mid Pf(p^\alpha A_\alpha)_i = 0 \quad \forall i = 1, \ldots, 7 \}$$

is called the Pfaffian subvariety of $\mathbb{P}^6$

defined by $p^\alpha A_\alpha$

Fact: It is a CY 3-fold

Local CY intersection
complete

see degrees are correct in each patch

p^α = homogeneous coordinates on toric variety

$A_\alpha = (2r+1) \times (2r+1)$ skew matrix

$$\langle \mathrm{Pf}((p^\alpha A_\alpha)_i) \rangle_{i=1 \dots 2r+1} \subseteq X$$

<u>GLSM</u> $G = U(1)^{N-1} \times U(2)$ or $SU(2)$

V has basis $\underbrace{p^1, \dots, p^N}_{\det^{-1} \text{ rep of } U(2)}, \ (\Phi^a_j)$ $a=1,2$

Φ_j fundamental of $U(2)$

<u>Example</u> $G = U(2)$

$W = \sum A_a^{jk} \epsilon_{ab} \Phi^a_j \Phi^b_k$ transforms like the det under $U(2)$

D-term <u>Vacua</u>: $\mu(P, \Phi) = \frac{1}{2}\left(-\sum |P|^2 + \sum \cancel{\det \Phi}\, \Phi^{\dagger a} \Phi^a\right)$

Notation Φ, P interchanged

$r<0$: $\vec{p} \neq \vec{0}$

$$\begin{array}{c} \text{vector bundle} \,/\, SU(2) \\ \downarrow \\ \mathbb{P}^6 \end{array}$$

Fix $\vec{p} \in \mathbb{P}^6$

consider $V_{\vec{p}}$ / $SU(2)$ vacua in theory

rank (

<u>F-term analysis</u> $(r<0 : \vec{p} \neq \vec{0})$

rank $(p^\alpha A_\alpha) = 2r \Rightarrow \left.\begin{array}{ll} 1 & \text{massless} \\ 2r & \text{massive} \end{array}\right\}$ $M \to \infty$ no vacua

rank $(p^\alpha A_\alpha) = 2r-2 \Rightarrow \left.\begin{array}{ll} 3 & \text{massless} \\ 2r-2 & \text{massive} \end{array}\right\}$ $M \to \infty$ unique vacuum state

Born-Oppenheimer approx

Classical vacua : Pfaffian locus

$\mathbb{P}^6 \quad 7_X 7$

$\mathbb{P}^{[a_1 \ldots]}$, $2r+1 \quad 2r+1$

~ 6 examples

<u>Optimistic Hope</u> All CY's could be realized this way