Morrison: Stability in Alg Geom + Wall Crossing Form

Thaddeus: Stable pairs, linear systems, the verlinde formula
alg-geom / 9210007

Nagao - Nakajima: Counting invariant of pervers chohevent sheaves and its wall-crossing

math. AG 0809.2992

(Coarse moduli space) $\leftrightarrow$ (Fine moduli space = moduli spaces w/ universal family)

Moduli spaces:

$\{\text{alg. objects}\} \longrightarrow V$

$\{\text{alg. objects + extra data}\} \longrightarrow X$

$G = \text{Aut}$

$V = X/G$

e.g. {hypersurfaces of fixed degree in $\mathbb{CP}^n$}

extra data = equation homog deg d, (n+1) vars $\longrightarrow$ $\overset{X}{\overset{\shortparallel}{\{\text{coeffs of eqn}\}}} = \mathbb{C}^N$

$G = GL(n+1, \mathbb{C})$ acts on $X = \mathbb{C}^N$

V "=" X/G

<u>issues</u>: 1) some orbits are <u>not</u> closed
2) some orbits lie in the closure of many many orbits

e.g. $\{0\} \subseteq \mathbb{C}^N$ is closure of every orbit

<u>Mumford</u>: $X_{\text{unstable}} = \bigcup (\text{very bad orbits})$

$X_{\text{semi-stable}}$ = non-unstable pts w/ non-closed orbits

X_{stable} = non-unstable pts w/ closed orbits

(choice of „linearization" can affect answer)

$$X_{\text{stable}}/G \subseteq \boxed{X_{\text{semistable}}/G}$$

(affine) typically projective variety (compact) e.g. elliptic curves in $\mathbb{CP}^2$

$$\{\text{curves in } \mathbb{CP}^2\} \setminus \{\text{bad ones}\} \Big/ Gl(3, \mathbb{C})$$

"throw away bad ones"

$\vdots$

Weierstraß-form $y^2 = x^3 + g_2 x + g_3$

$$(g_2, g_3) \mapsto (\lambda^2 g_2, \lambda^3 g_3)$$

action of $\mathbb{C}^*$

cases completed:

$n = 1$ ✓

$n = 2$ ✓

$n = 3$ $d \leq 4$

e.g. $\mathbb{C}^*$ acts on $\mathbb{C}^4 = \{(x, y, z, t)\}$

$$\lambda : (x, y, z, t) \mapsto (\lambda x, \lambda y, \lambda^{-1} z, \lambda^{-1} t)$$

orbits of concern:

- $(x, y, 0, 0)$ has $(0,0,0,0)$ in its closure
- $(0, 0, z, t)$ has $(0,0,0,0)$ in closure

choice 0: keep all orbits

$\{(x,y,0,0)\} \sim \{(0,0,z,t)\} \sim \{(0,0,0,0)\}$

choice −: remove $\{(x,y,0,0)\}$ before taking quotient

$\{(0,0,z,t)\} \longrightarrow [z:t] \in \mathbb{P}^1$

choice +: remove $\{(0,0,z,t)\}$ before taking quotient

$\{(x,y,0,0)\} \longrightarrow [x:y] \in \mathbb{P}^1$

Alternative (Ness, Kirwan, 1980's
Guillemin, Sternberg late 1980's)

$G = G_{\mathbb{C}} \supseteq G_{\mathbb{R}}$ = compact grp whose cplx. is G

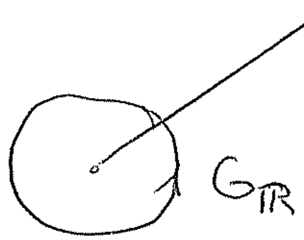
(in example above: $G_{\mathbb{R}} = U(1)$)

X/G:

symplectic reduction: $\mu : X \longrightarrow \mathrm{Lie}(G_{\mathbb{R}})^*$

X/G "=" $\mu^{-1}(a)/G_{\mathbb{R}}$ fixed $a \in \mathrm{Im}\,\mu$

$\mathbb{C}^*$



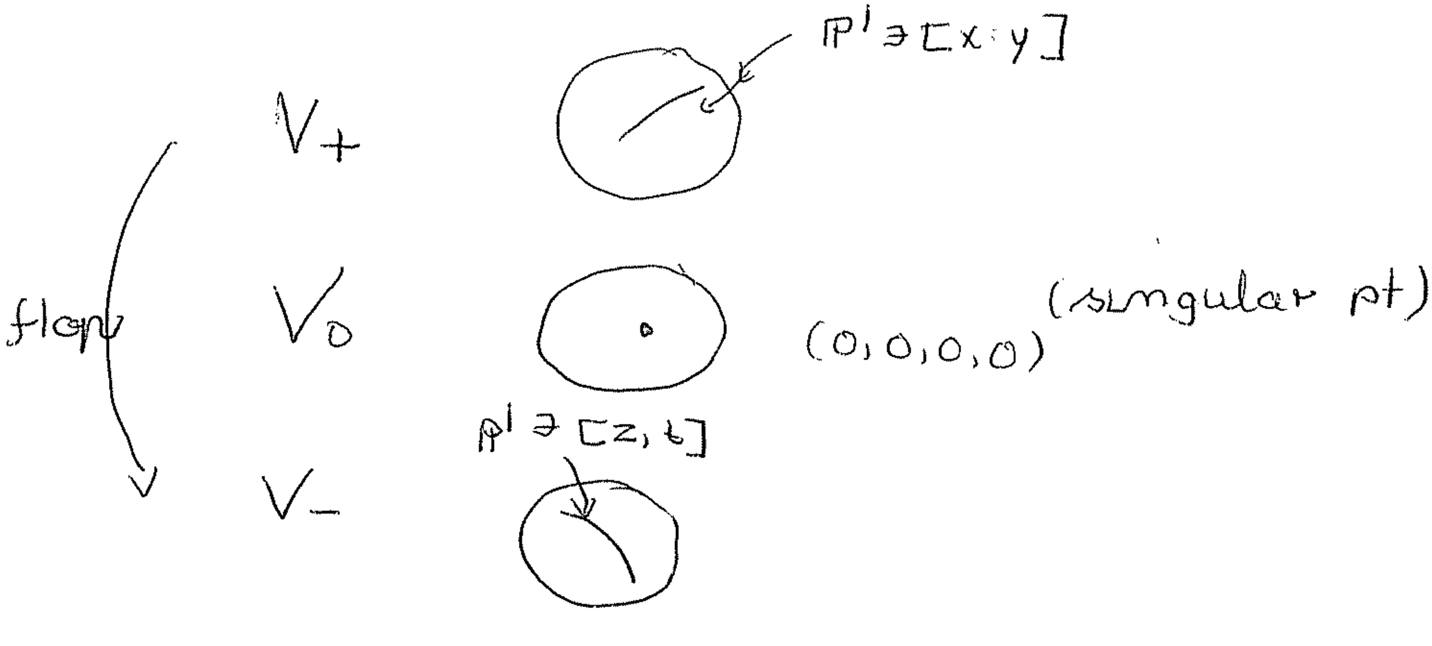
in example:

$$\mu(x,y,z,t) = \frac{1}{2}\left(|x|^2+|y|^2-|z|^2-|t|^2\right)$$

- $\underline{a>0}$: exclude $x=y=0$

"choice +" $= \left(|x|^2+|y|^2-|z|^2-|t|^2=a\right) / U(1)$

- $\underline{a<0}$: exclude $z=t=0$
- $\underline{a=0}$: no exclusion



$\mathbb{C}^*$ $\quad \lambda x, \lambda y, \lambda^{-1} z, \lambda^{-1} t$

$$\underset{\alpha}{\underset{\parallel}{xz}}, \quad \underset{\beta}{\underset{\parallel}{xt}}, \quad \underset{\gamma}{\underset{\parallel}{yz}}, \quad \underset{\delta}{\underset{\parallel}{yt}}$$

$$\alpha\delta = \beta\gamma$$

Moduli of vector bundles (on curves)

$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

Q: do there exist subbundles of E high degree ?
(if so, semi-stable or unstable)

rank $E = 2$

$$\deg L \leq \tfrac{1}{2} \deg E$$

$\deg E = \deg(\Lambda^2 E)$
$\Lambda^2 E$ is line-bdl on X

$\deg L \leq \frac{1}{2} \deg E \quad \forall L$ semistable
$\deg L < \frac{1}{2} \deg E \quad \forall L$ stable
$\deg L > \frac{1}{2} \deg E$ for one L unstable

$E \rightsquigarrow Gr(\mathbb{C}^2 \subseteq \mathbb{C}^N)$
Gl (N-tuples of sections)

$E = \mathcal{O} \oplus L$

Phenom:

$\exists$ family of rk-2 v. bdls over $\mathbb{CP}^1$ param. by t, s.t.

$$E_t \cong \mathcal{O}(a) \oplus \mathcal{O}(b) \qquad t \neq 0$$
$$E_0 \cong \mathcal{O}(a+1) \oplus \mathcal{O}(b-1) \qquad t = 0$$

Thaddeus

$$\left(\begin{array}{c} E \\ \downarrow \\ X \end{array}, \ \varphi \in H^0(E) \setminus \{0\} \right)$$

Various possible stability cond:

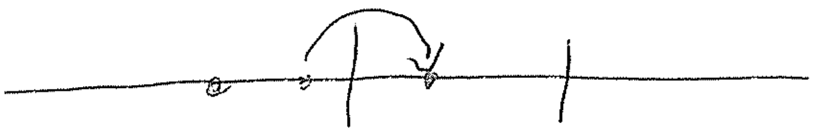
stability param $\sigma \in \mathbb{Q}_+$

(E, φ) is σ-semi(stable)

if $\forall\ L \subseteq E$
- $\deg L \underset{(-)}{\leq} \frac{1}{2} \deg E - \sigma$ if $\varphi \in H^0(L)$
- $\deg L \underset{(-)}{\leq} \frac{1}{2} \deg E + \sigma$ if $\varphi \notin H^0(L)$

Each of these leads to V_σ via GIT

Thaddeus showed how to blow up or down V_σ to get to $V_{\sigma'}$



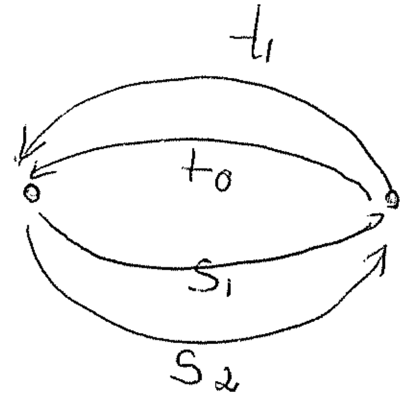
Nagao – Nakajima

sheaves living on V_+, V_0, V_-

Van den Bergh $\quad D^b(\mathrm{Coh}(V_+)) \cong D^b(\mathrm{mod}-\mathcal{A})$

$$\begin{array}{c} \| \wr \\ D^b(\mathrm{Coh}(V_-)) \end{array}$$

- $\mathcal{A}$ non-com ring
  mod-$\mathcal{A}$: modules over $\mathcal{A}$
  stability conditions $\rightsquigarrow$ moduli space
- $\mathcal{A}$ is an algebra derived from gauge theory description of D-branes on V_0
  (Klebanov, Witten, M-Plesser)



gauge grp G_0, G_1 (for each vertex)

V_0, V_1 vct - spaces

$S_1, S_2 \in \mathrm{Hom}(V_0, V_1)$

$T_1, T_2 \in \mathrm{Hom}(V_1, V_0)$

superpotential: $tr(W)$

$$W = \cancel{tr}\,(t_2 s_2 t_1 s_1 - t_1 s_2 t_2 s_1 + \text{cyclic})$$

$$\mathcal{A} = \begin{pmatrix}\text{algebra generated} \\ \text{by paths} \\ \text{in diagram}\end{pmatrix} \Big/ \left(\frac{\partial W}{\partial x} \right)$$

$$\left(\frac{\partial W}{\partial x} \right) = (s_2 t_1 s_1 - s_1 t_1 s_2, \ldots)$$

Representation space

given $\quad n_0, n_1 = \dim V_0, \dim V_1$

$S_1, S_2 \in \mathrm{Hom}(\mathbb{C}^{n_0}, \mathbb{C}^{n_1})$

$T_1, T_2 \in \mathrm{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_0})$

$S_1 T_1 S_2 = S_2 T_1 S_1$ etc

isomor.