

The Geometric Langlands Conjecture I:

Arithmetic & Geometry
Abelian & non-Abelian

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UCSB, 5/12/2008

- * Arithmetic & Geometry
- * ~~Abelian~~ Class Field Theory
- * Geometric Langlands
- * Hecke transforms
- * Abelianization.

next:

- QFT
- non-abelian Hodge theory
- Higgs bundles & Koszul
- The classical limit

* GLC: roots, chars, weights

• Bun, Loc, D^b

- Hecke correspondences, (affine) Grassmannians
- Hecke operators, automorphic sheaves
- stacks & gerbes

* Examples: $GL(n)$, $GL(1)$.

- Higgs bundles, Hitchin system, abelianization
- Spectral & canonical covers

* Ancient approach: eigen sheaves of abelianized
Higgs

- \rightsquigarrow duality between Higgs "Higgs"

* Non abelian Hodge theory

- compact vs. non-compact
- stable, unstable, & wobbly bundles

* Koszul resolutions & Higgs bundles

- Recent physics input

* Classical limit: definition

• conjecture

• theorem

proof:

- Duality along the base
- " " " fibers

- Transcendental ingredient: Hodge theory
- Duality for gerbes.

The Geometric
Langlands Conjecture II:
Algebra & Analysis
Classical & Quantum

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UCSB, 5/13/2008

- * Review: GLC
- * abelianization
- * non-abelian Hodge theory
- * QFT
- * the classical limit

Geometric Langlands Conjecture

∃ natural equivalence of categories:

$$c: D^b(\text{Loc}) \xrightarrow{\sim} D^b({}^L\text{Bun}, \mathcal{D})$$

sending structure sheaves of points V in Loc to automorphic \mathcal{D} -modules on ${}^L\text{Bun}$:

$$H^m(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \otimes \rho^m(V)$$

Notation:

C ; $G, T, \mathfrak{g}, \mathfrak{t}$; ${}^L G, {}^L T, \text{etc.}$

roots $_{\mathfrak{g}}$	\subset char $_{\mathfrak{g}}$	\subset weights $_{\mathfrak{g}}$
coroots $_{\mathfrak{g}}$	\subset cochar $_{\mathfrak{g}}$	\subset coweights $_{\mathfrak{g}}$
 roots $_{{}^L\mathfrak{g}}$	 char $_{{}^L\mathfrak{g}}$	 weights $_{{}^L\mathfrak{g}}$

$\text{Bun}, {}^L\text{Bun}$: mod. sp. of semistable principal $G, {}^L G$ bundles on C

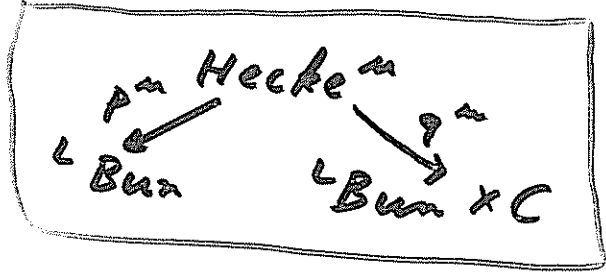
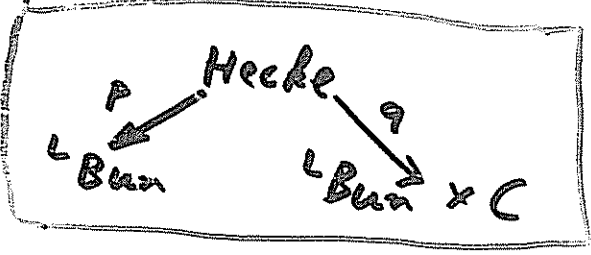
$\text{Loc}, {}^L\text{Loc}$: mod. sp. of semistable $G, {}^L G$ local systems $V = (V, \sigma)$ on C

Bun, Loc etc.: the corresponding moduli stacks

Hecke correspondence :=

moduli space of quadruples (V, V', x, β) :

- V, V' : principal G -bundles on C
- $x \in C$
- $\beta: V|_{C, x} \xrightarrow{\sim} V'|_{C, x}$ an isomorphism



$\lambda \in \text{char}_{L G}^+ = \text{cochar}_G^+$ dominant characters of $L G$

$\rho = \rho_\lambda$ irrep of $L G$ with h.w. = λ

$\mu \in \text{cochar}_{L G}^+ = \text{char}_G^+$ dom. char. of G

$$\text{Hecke}^\mu := \left\{ (V, V', x, \beta) \mid \begin{array}{l} \forall \lambda \in \text{char}_{L G}^+ \cup \{ \rho = \rho_\lambda \} \\ \rho(\beta): \rho(V) \rightarrow \rho(V') \otimes \mathcal{O}_C(\mu, \lambda, x) \end{array} \right\}$$

- Fibers of p, q : ∞ -dim'l ind-schemes
 $q^{-1}(V', x) = \text{affine Grassmannian}$

Hecke^m \subset Hecke: f.d. closed subscheme

- Hecke^m is smooth $\Leftrightarrow \mu$ minuscule weight
- Hecke = \varinjlim_m Hecke^m
- p^m, q^m : proper, loc. trivial fibrations.

Hecke operators = integral transforms

$$H^{\alpha} : D^b({}^L\text{Ban}, \mathcal{D}) \rightarrow D^b({}^L\text{Ban} \times \mathbb{C}, \mathcal{D})$$
$$m \mapsto q_! (p^{L*} m \otimes \mathcal{I}(\alpha)) [d_m]$$

Similarly: H_x^{α} , for $x \in \mathbb{C}$.

⇒ Hecke algebra A generated by the H_x^{α} .
Abelian.

A \mathcal{D} -module m on ${}^L\text{Ban}$ is a Automorphic sheaf / Hecke eigen sheaf
w/ eigenvalue $\mathbb{V} = (V, \rho)$ a G -loc. sys. on \mathbb{C} if:
$$H^{\alpha}(m) = m \boxtimes \rho^{\alpha}(\mathbb{V})$$
 or.

With this setup, GLC makes sense,
except...

Discrepancy: ${}^L\text{Bun}$ is disconnected.

$$\pi_0({}^L\text{Bun}) = H^2(C, \pi_1({}^L\mathbb{G})) = \pi_1({}^L\mathbb{G}) = \mathbb{Z}/6\mathbb{Z}^A \Rightarrow$$
$$D^b({}^L\text{Bun}, \mathcal{D}) = \coprod_{\gamma \in \pi_1({}^L\mathbb{G})^A} D^b({}^L\text{Bun}^\gamma, \mathcal{D})$$

while Loc is irreducible $\Rightarrow D^b(\text{Loc})$ indecomposable.

So: replace Loc by

$\text{Loc} :=$ moduli stack of semistable G -local systems

Loc is an alg. stack w/ coarse moduli space = Loc .

\exists open substack Loc^{rs} of "regularly stable" local systems

$\text{Loc}^{\text{rs}} \rightarrow \text{Loc}^{\text{rs}} : \text{banded } \mathbb{Z}/6\mathbb{Z}\text{-gerbe}$

$$D^b(\text{Loc}^{\text{rs}}) = \coprod_{\gamma \in \mathbb{Z}/6\mathbb{Z}^A} D^b(\text{Loc}^{\text{rs}}, \gamma).$$

Example: $G = {}^L G = GL(n)$

$V = (V, \rho) = \text{rank } n \text{ loc. sys. } / C$

Hecke algebra A is generated by H^i :

Hecke $i = \{(V, V', \alpha) \mid V \subset V' \subset V(\alpha), i = \mathcal{L}(V'/V)\}$

fiber = $Gr(i, n)$.

Example: $G = GL(1)$, $Bun = Pic$

$H^1: C \times Pic^d(C) \rightarrow Pic^{d+1}(C)$

$\alpha, L \mapsto L(\alpha)$ abel. Jacobi.

$V = (L, \rho)$

$c(V) =$ the unique loc. sys. on $Pic(C)$
restricting to (L, ρ) .

$(\pi, (Pic^d C) = \pi, (C) / C, \mathbb{Z}.)$

$$\text{Higgs} = \text{Higgs}_{C, E} = \{ (V, \varphi) \mid \varphi: V \rightarrow V \otimes \omega_C \}$$

Hitchin map:

$$h: \text{Higgs}_{C, E} \rightarrow B_{C, E} := H^0(C, (\omega_C \otimes \mathcal{E})/w) = \bigoplus_{i=1}^r H^0(C, \omega_C \otimes \mathcal{L}_i)$$

$$(V, \varphi) \mapsto \text{"}\varphi \text{ mod } w\text{"} \leftrightarrow (I_1(\varphi), \dots, I_r(\varphi))$$

General covers:

$$\begin{array}{ccc} C \hookrightarrow \tilde{E} & \rightarrow & \text{Tot}(C, \omega_C \otimes \mathcal{E}) \\ \downarrow & & \downarrow \\ C \times B & \rightarrow & \text{Tot}(C, (\omega_C \otimes \mathcal{E})/w) \end{array}$$

$$h^{-1}(L) \sim \text{Princ}(C/E)$$

$h: \text{Higgs} \rightarrow B$ integrable system

$$\text{Higgs} \supset T^*B_{un}$$

$\nu_n \ni$ associated spectral covers \tilde{E}_n^* , $\tilde{E}_n^* \rightarrow C \times B$.
 e.g. $G = \text{GL}(n) \Rightarrow \text{deg}(\tilde{E}_n^*/C) = n$.

[De Gaitsgory '00]: $\pi: \tilde{E} \rightarrow C \times B$ determines an abelian group scheme T over $C \times B$. $h: \text{Higgs} \rightarrow B$ is a principal homogeneous stack over the ab. group stack Tor_T of T -torsors.

= "abelianization".
 (Hitchin, Atiyah, Beilinson-Kazhdan, Faltings, ...)

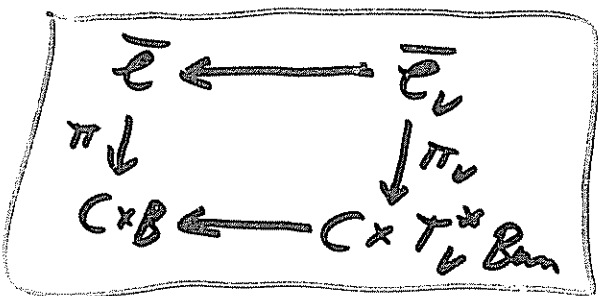
Ancient approach

$(G = GL(n))$

$V = (V, \sigma) \Rightarrow$

L on $\bar{E}_V, \pi_V^* L \approx V \otimes \omega_C^{- (n-1)/2}$

σ induces hol. conn. on L



(Depends on $\sqrt{\omega_C}$: use $\text{Ran}_\sigma \approx \pi_\sigma^* \omega_C^{\otimes (n-1)}$
 The natural induced connection σ' has residue $= \frac{1}{2}$
 along Ran_σ . Set $\sigma = \sigma' + \frac{1}{2} d \log f_\sigma$, where
 $(f_\sigma) = \text{Ran}_\sigma + 2(n-1) \pi_\sigma^* f$, f a section of $\sqrt{\omega_C}$)

Use case $G = GL(1)$: spread (L, σ) to $(\tilde{V}, \tilde{\sigma})$ on
 $\text{Pic}(\bar{E}_V / C \times T_V^* B_m) =: \widetilde{\text{Higgs}} = \text{Higgs}_B \times_B T_V^* B_m$

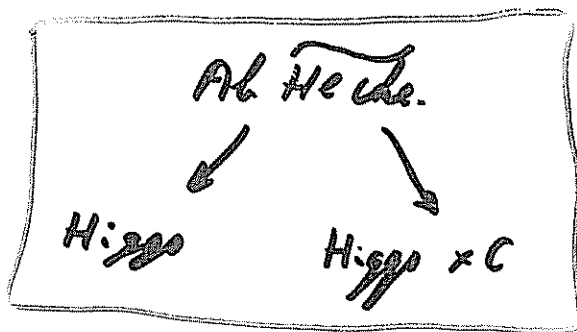
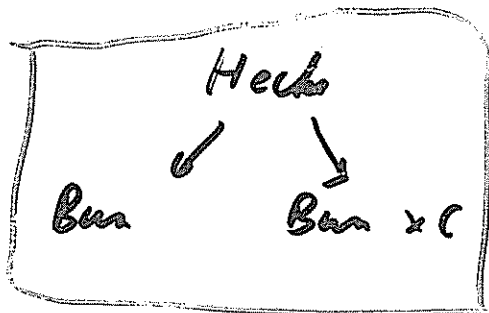
Now push to $\text{Higgs} \Rightarrow$

$(\tilde{V}, \tilde{\sigma})$ on Higgs , zero loc. eqs along fibers over B .

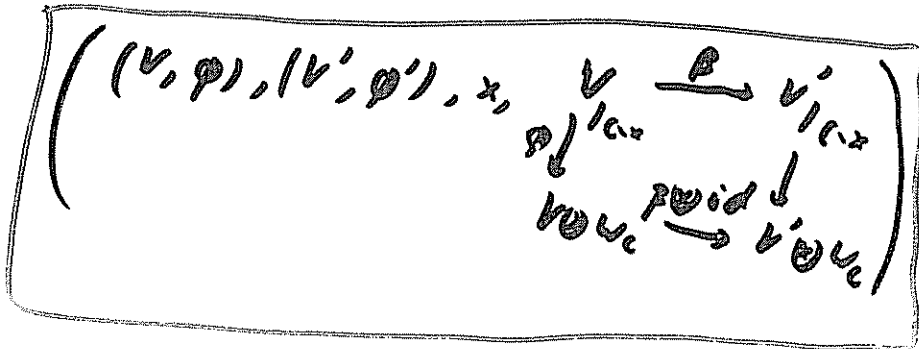
Fiber of \tilde{V} at $(E, \psi) \in \text{Higgs}$: $(E, \psi) \leftrightarrow (\bar{E}_\psi, M_\psi) =: \bar{E}_\psi$

$\tilde{V}_{(E, \psi)} = \bigoplus_{\substack{\varphi \in T_V^* B_m \\ h(\varphi) = h(\psi)}} (L_\varphi, M_\psi) \bar{E}_\psi$

↑ fiber of Poincare bundle



$(V, V', x, \beta) \dots$



$G_r(i, n)$

finite collection of φ_x -eigenspaces

\overline{V} has an automorphic property w.r.t. Hecke.

To state this for all G , need: a duality between $Higgs_G$ and $Higgs_{LG}$.

To finish:

- * Average over all $\varphi \in T^* Bun$?
- * Or: deform
- * Or: use Simpson's.

Non abelian Hodge theory

(Simpson, Corlette, ...)

$$\text{Higgs}^{\circ}(M) \longleftrightarrow \text{Loc}(M)$$

M compact, Kähler.

Our case:

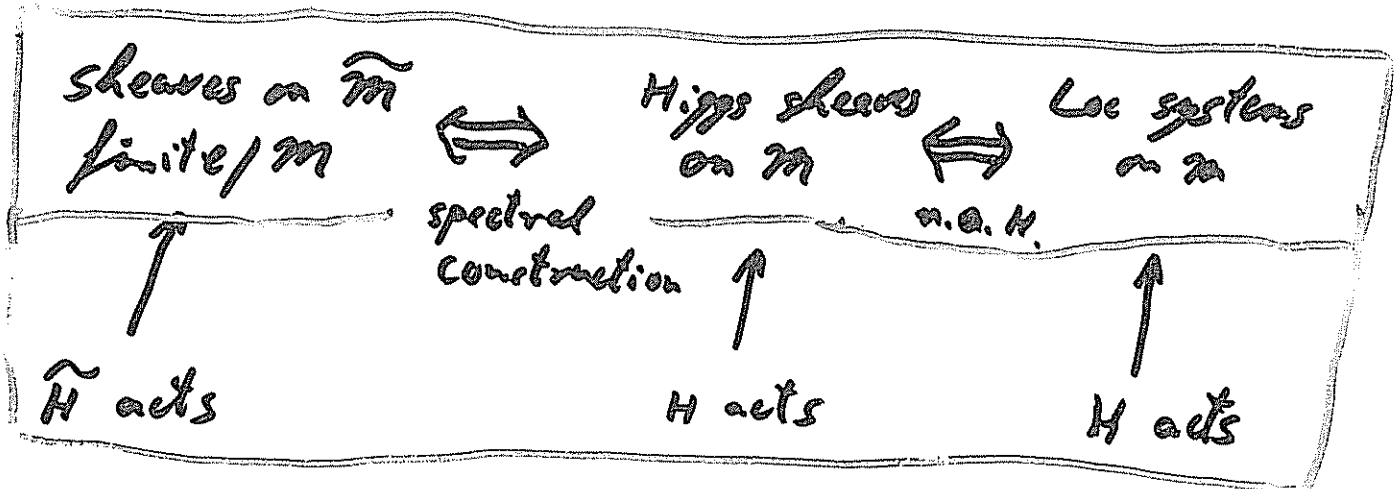
$$M = \text{Bun}$$

$$\tilde{M} = \text{Higgs} \sim T^* \text{Bun}$$

$$H \subset M \times M \quad \text{Hecke}$$

$$\tilde{H} := N^*_{H/M \times M} \subset \tilde{M} \times \tilde{M}$$

Equivalences:



In general: actions are compatible

Our case:

$$\text{Key: } \tilde{H} = \text{abelianized Hecke.}$$

Start: \bar{V} is an eigensheaf of \bar{H} on \bar{M}

\Leftrightarrow

~~\bar{V}~~ eigen Higgs sheaf on M

$\stackrel{?}{\Leftrightarrow}$

eigen local system on M ?

problem: V defined only on Z open in M
blows up along divisors:

| * branch divisor

| * non-very-stable: "wobbly"

Status:

Simpson: OK on curves

Beauchamp-Biquard

...

Mochizuki: OK w/ NCD, conditions.

(Hertling, Sabbah, ...)

Jost-Yang-Zuo

Key issue: geometry of wobbles.

$E \rightarrow C$ v.B. is very stable if:

$$\boxed{\begin{array}{l} \varphi: E \rightarrow E \otimes K_C \\ \varphi \text{ nilpotent} \end{array}} \Rightarrow \boxed{\varphi = 0}$$

very stable \Rightarrow stable

nilpotent cone

$$\boxed{\begin{array}{c} N \subset T^* \text{ Bun} \\ \downarrow \\ \text{Bun} \end{array}}$$

$$N = \bigcup_i N^* S_i \quad S_i \subset \text{Bun} \quad (S_0 = \text{Bun})$$

$$\text{Non-very-stable} = \bigcup_i S_i$$

wobbly

\Rightarrow locus where automorphic Higgs sheaf on Bun blows up

$E \rightarrow X$ rank n v.b.
 $\alpha \in H^0(X, E)$ $Y_\alpha = \{\alpha = 0\} \subset X$ smooth

\Rightarrow Koszul $(\pi^* E, \pi^* \alpha)$
 or: $(\pi^* E^\vee, i_\alpha)$

exact when $\alpha \neq 0 \dots \Rightarrow$

$(\pi^* E, \pi^* \alpha) \cong i_{Y_\alpha} \pi^* F$

$0 \rightarrow N^\vee \rightarrow E|_{Y_\alpha} \rightarrow F \rightarrow 0$ $i_{Y_\alpha}: Y_\alpha \hookrightarrow X$

can $E = \Omega_X^1$:

$H^k(X, (\pi^* T_X, i_\alpha)) \cong \bigoplus_{a+b=k} H^a(Y, \pi^* T_Y \oplus \det N)$
 $H^k(X, (\pi^* \Omega_X, \pi^* \alpha)) \cong \bigoplus_{a+b=k} H^a(Y, (\Omega_Y^b, 0))$

more generally:

(V, \mathcal{R}) Higgs bundle / X
 $\Leftrightarrow (C, L)$, $C \subset T^*X$ spectral cover
 $L \in \text{Pic}(C)$ spectral sheaf
 $X_0 := C \cap T^*X$

\Rightarrow
 $H_{\text{Higgs}}^*(X, (V, \mathcal{R})) = H^*(X, \nu \otimes \nu^* \otimes \Omega^1 \otimes \nu \otimes \Omega^2 \otimes \dots)$
 $= H^*(X_0, L \otimes \pi^* V)$

Apply this to H, \tilde{H} on \mathcal{M} , where

$X = \text{fiber of } H \sim \text{Grassmannian}$
 $X_0 = \text{fiber of } \tilde{H} \sim \text{finite Grassmannian}$
(= fixed points of ρ)

\Rightarrow H action on H -sheaves on \mathcal{M} is
compatible with
 \tilde{H} action on sheaves on $\tilde{\mathcal{M}}$.



Recent physics input:

Kapustin - Witten

Gukov - Witten

Frenkel - Witten

interprets GLC in QFT

A-branes \Leftrightarrow B-branes

"mirror symmetry"

Wilson ops
+ Hooft ops
.....

arrives at similar conclusion:

Abelianized GLC

=

classical limit of GLC

but also

\Leftrightarrow Full GLC

Via non-abelian Hodge on
open manifolds.

classical limit

λ -connections: $\mathcal{O}(S) = \mathcal{O}_S \times \lambda d/$

$$D: V \rightarrow V \otimes \mathcal{O}'$$

$\lambda=1$: connection

$\lambda \neq 0$: ditto

$\lambda=0$: Higgs field.

Simplex: diffeomorphism.

As $\lambda \rightarrow 0$:

($X = \text{Bun } G, \mathcal{L}_G$)

$$\text{Loc}_X \rightarrow \text{Higgs}_X$$

\mathcal{D}_X -modules $\rightarrow \text{Sym}^* T_X$ -modules = coherent sheaves on T_X^* .

classical limit of GIC:

$$D_{\text{coh}}^b(\text{Higgs}_{G, \mathcal{L}_G}^0) \cong D_{\text{coh}}^b(\text{Higgs}_{G, \mathcal{L}_G})$$

note: need to understand $\lim_{\lambda \rightarrow 0} \mathcal{IC}^n$ of Artinkin.

Classical limit conjecture:

\exists natural equivalence of categories

$$c: D^b(\mathcal{Higgs}) \xrightarrow{\sim} D^b({}^L\mathcal{Higgs})$$

inducing

$$c^\circ: D^b(\mathcal{Higgs}^\circ) \xrightarrow{\sim} D^b({}^L\mathcal{Higgs}^\circ)$$

c° sends str. sheaves of points in \mathcal{Higgs}° to Hecke eigen sheaves:

$$(V, \rho) \in \mathcal{Higgs}^\circ \Rightarrow$$

$$H^m(c^\circ(V, \rho)) \cong c^\circ(V, \rho) \boxtimes (\rho^*(V), \rho^*(\rho))$$

D & Pantev 06/04/617: true over $B \cdot \Delta$.

Δ = discriminant, parametrizes singular conical covers.

Underlying geometry: $\mathcal{Higgs}, {}^L\mathcal{Higgs}$ are dual integrable systems.

Hausel & Thaddeus: cases $GL(=1), SL(=1)$.

Hitchin: G_2

Arinkin, Ngo: some info / Δ

Ngo: \Rightarrow "Fundamental Lemma"

steps

Duality along the base:

$$\exists \text{ isomorphism } \ell: \mathfrak{B} \xrightarrow{\sim} {}^L\mathfrak{B}$$

$$\ell(\alpha) = \alpha$$

$$\text{lifts to: } \ell: \tilde{\mathfrak{E}} \xrightarrow{\sim} {}^L\tilde{\mathfrak{E}}$$

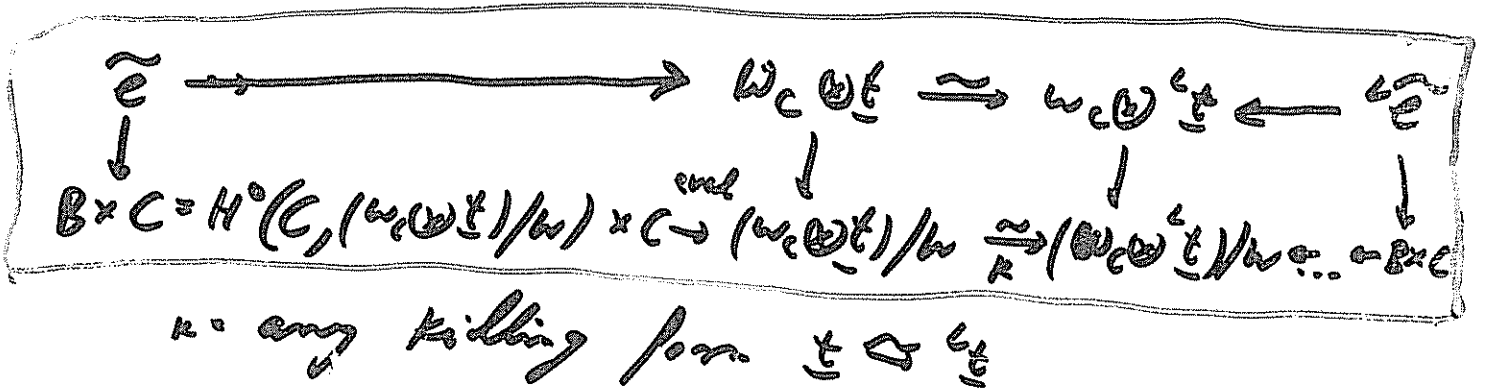
interchanges short + long roots.

A, D, E, F, G self dual algebras \Rightarrow

$$\ell: \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}$$

$$A, D, E: \ell = \text{id}$$

$$F, G: \ell \neq \text{id}$$



Duality along fibers

$$\begin{aligned} T_C &= \Lambda \otimes C^* \\ \Lambda &= \text{cotangent } C \end{aligned}$$

[D+ Gaiitsgorj]:

$$p: \tilde{C} \rightarrow C \quad \text{universal} \quad \Rightarrow$$

$$\begin{aligned} \tilde{T} &:= p_* (\Lambda \otimes \theta_{\tilde{C}}^*) \\ T &:= \{t \in \tilde{T} \mid \alpha(t)|_{D^\alpha} = 1 \text{ } \forall \text{ root } \alpha \text{ of } \mathcal{D}\} \end{aligned}$$

$$\alpha: T \rightarrow C^* \quad \alpha^\vee: C^* \rightarrow T$$

$D^\alpha \subset \tilde{C}$: fixed divisor for reflection p^α

$$T^0 := \text{connected component of } T$$

$T^0 \subset T \subset \tilde{T}$ sheaves of ab. gps. on C

[DG]: $L^\vee(\mathcal{D})$ is a torsor over $H^1(T)$

Real versions: $\theta_{\tilde{C}}^* \leftrightarrow S'$

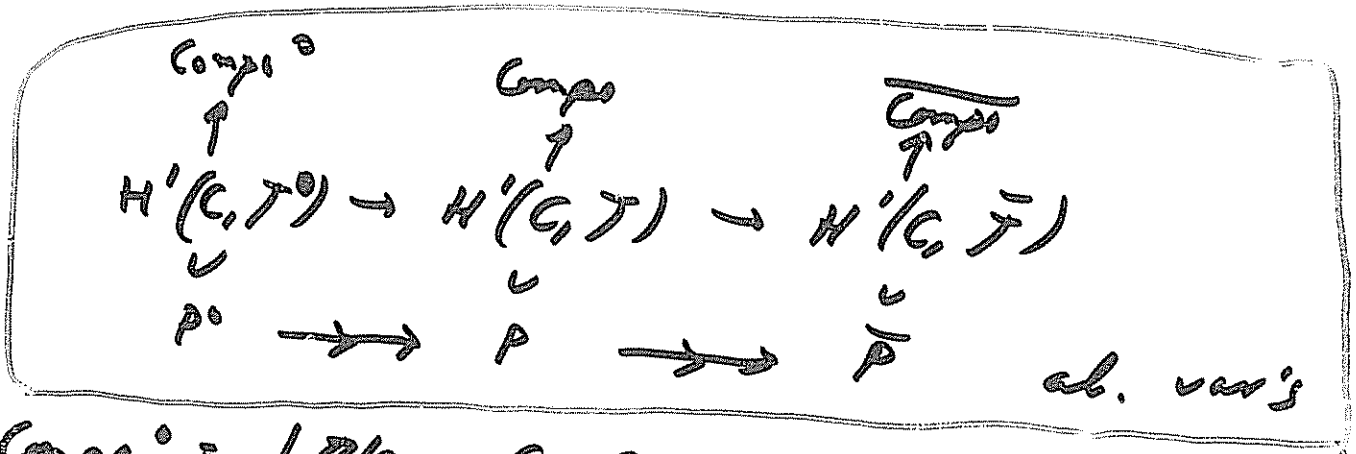
$$\begin{aligned} \tilde{T}_{R,S} &= \{\lambda \otimes \tau \mid \alpha^\vee(\tau^{(\alpha, \lambda)}) = 1 \text{ in } T_R\} \\ \cup \\ T_{R,S} &= \{\lambda \otimes \tau \mid \tau^{(\alpha, \lambda)} = 1 \text{ in } S'\} \\ \cup \\ T^0_{R,S} &= \{\lambda \otimes \tau \mid (\alpha, \lambda) = 0 \text{ in } \mathcal{D}\} \end{aligned}$$

Hodge theory from [ODP 2005]:

holo + real sheaves have same H^1

reason: Hodge theory \Rightarrow finite her, coher
but the cone is a complex of \mathbb{R} v.s. \mathbb{C} .

\Rightarrow topological calculations.



$$\text{Comp}^0 = \begin{cases} \mathbb{R}/2 & G = Sp(N) \\ \pi_1(G) & \text{else} \end{cases}$$

$$\text{Comp} = \pi_1(G) \text{ always}$$

so:

Comp's of Hitchin fiber \leftrightarrow
Comp's of Higgs.

$$\text{Key: } \text{Comp}^0 = H^1(u, \mathcal{A}^0) \text{ for } \\
 = \left(\frac{(\mathcal{A}^0)^\vee}{(1-P_1, \dots, 1-P_r) \mathcal{A}^0} \right) \text{ for}$$

$$\text{coker}(P) = \text{coker}(P)^\vee, \quad \overline{P} = \text{cp.}$$

Duality for Higgs gerbes

Higgs is a braided \mathbb{Z} -gerbe

= over B , Higgs is a torsor over
the comm. gp. stack $\text{Tor}_\mathbb{Z}$.

\exists Hitchin section $B \rightarrow \text{Higgs} \Rightarrow$
it's the trivial torsor.