# The Geometric Langlands Conjecture I:

## Arithmetic & Geometry
## Abelian & non-Abelian

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* Arithmetic & Geometry
* Abelian Class Field Theory
* Geometric Langlands
* Hecke transforms
* Abelianization

next:

- QFT
- non-abelian Hodge theory
- Higgs bundles + Koszuls
- The classical limit

* GLC: • roots, chars, weights
  • Bun, Loc, $D^b$

• Hecke correspondences, (affine) Grassmanians
• Hecke operators, automorphic sheaves
• stacks & gerbes

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* Examples: $GL(n)$, $GL(1)$.

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• Higgs bundles, Hitchin system, abelianization
• Spectral & cameral covers

* Ancient approach: eigensheaves of abelianized heckes
• ⇝ duality between Higgs ${}^L$Higgs

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* non abelian Hodge theory
• compact vs. non-compact
• stable, unstable, & wobbly bundles

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* Kosjul resolutions & Higgs bundles
• Recent physics input

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* Classical limit: • definition
  • conjecture
  • theorem

proof:
• Duality along the base
• " " " fibers
• Transcendental ingredient: Hodge theory
• Duality for gerbes.

# The Geometric Langlands Conjecture II: Algebra & Analysis Classical & Quantum

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* Review: GLC
* abelianization
* Non-abelian Hodge theory
* QFT
* The classical limit

# Geometric Langlands Conjecture

$\exists$ natural equivalence of categories:

$$\boxed{c : D^b(\mathcal{L}oc) \xrightarrow{\sim} D^b({}^L\mathcal{B}un, \mathcal{D})}$$

sending structure sheaves of points $\mathbb{V}$ in $\mathcal{L}oc$ to automorphic $\mathcal{D}$-modules on ${}^L\mathcal{B}un$:

$$\boxed{H^\mu(c(\mathcal{O}_{\mathbb{V}})) = c(\mathcal{O}_{\mathbb{V}}) \boxtimes \rho^\mu(\mathbb{V})}$$

<u>Notation</u>:

$C$ ; $G, T, \mathfrak{g}, \underline{t}$ ; ${}^LG, {}^LT$, etc.

| | | |
|---|---|---|
| $\text{roots}_{\mathfrak{g}}$ $\subset$ | $\text{char}_G$ $\subset$ | $\text{weights}_{\mathfrak{g}}$ |
| $\text{coroots}_{\mathfrak{g}}$ $\subset$ <br> $\parallel$ <br> $\text{roots}_{{}^L\mathfrak{g}}$ | $\text{cochar}_G$ $\subset$ <br> $\parallel$ <br> $\text{char}_{{}^LG}$ | $\text{coweights}_{\mathfrak{g}}$ <br> $\parallel$ <br> $\text{weights}_{{}^L\mathfrak{g}}$ |

$Bun$, ${}^LBun$ : mod. sp. of semistable principal $G$, ${}^LG$ bundles on $C$

$Loc$, ${}^LLoc$ : mod. sp. of semistable $G$, ${}^LG$ local systems $\mathbb{V} = (V, \nabla)$ on $C$

$\mathcal{B}un$, $\mathcal{L}oc$ etc.: the corresponding moduli stacks

$\boxed{\text{Hecke correspondence}}$ :=

moduli space of quadruples $(V, V'; x, \beta)$:

- $V, V'$: principal ${}^L G$-bundles on $C$
- $x \in C$
- $\beta: V|_{C \setminus x} \xrightarrow{\sim} V'|_{C \setminus x}$ an isomorphism

$$\boxed{\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \times C \end{tikzcd}} \qquad \boxed{\begin{tikzcd} & \mathrm{Hecke}^{\mu} \arrow[dl, "p^{\mu}"'] \arrow[dr, "q^{\mu}"] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \times C \end{tikzcd}}$$

$\lambda \in \mathrm{char}^+_{{}^L G} = \mathrm{cochar}^+_G$ dominant character of ${}^L G$

$\rho = \rho_\lambda$ irrep of ${}^L G$ with h.w. $= \lambda$

$\mu \in \mathrm{cochar}^+_{{}^L G} = \mathrm{char}^+_G$ dom. char. of $G$

$$\boxed{\mathrm{Hecke}^{\mu} := \left\{ (V, V'; x, \beta) \,\middle|\, \begin{array}{l} \forall \lambda \in \mathrm{char}^+_{{}^L G} \text{ w/ } \rho = \rho_\lambda, \\ \rho(\beta): \rho(V) \to \rho(V') \otimes \mathcal{O}(\langle \mu, \lambda \rangle x) \end{array} \right\}}$$

- Fibers of $p, q$: $\infty$-dim'l <u>ind-schemes</u>
  $q^{-1}(V', x)$ = affine Grassmannian
- $\boxed{\mathrm{Hecke}^{\mu} \subset \mathrm{Hecke}: \text{ f.d. closed subscheme}}$
- $\mathrm{Hecke}^{\mu}$ is smooth $\Leftrightarrow$ $\mu$ minuscule weight
- $\mathrm{Hecke} = \varinjlim_{\mu} \mathrm{Hecke}^{\mu}$
- $p^{\mu}, q^{\mu}$: proper, loc. trivial fibrations.

$\boxed{\text{Hecke operators}}$ = integral transforms

$$\boxed{\begin{aligned} H^{\lambda} : D^b({}^L\mathrm{Bun}, \mathcal{D}) &\longrightarrow D^b({}^L\mathrm{Bun} \times C, \mathcal{D}) \\ \mathcal{M} &\mapsto q^{\lambda}_! \, (p^{\lambda *}\mathcal{M} \otimes \mathcal{I}(\lambda)) \, [d_\lambda] \end{aligned}}$$

Similarly: $H^{\lambda}_x$, for $x \in C$.

$\Rightarrow$ <u>Hecke algebra $A$</u> generated by the $H^{\lambda}_x$.

Abelian.

A $\mathcal{D}$-module $\mathcal{M}$ on ${}^L\mathrm{Bun}$ is a <u>Hecke eigensheaf</u> (<u>Automorphic sheaf</u>) w/ eigenvalue $\mathbb{V} = (V, \nabla)$ a $G$-loc. sys. on $C$ iff

$$\boxed{H^{\lambda}(\mathcal{M}) = \mathcal{M} \boxtimes \rho^{\lambda}(\mathbb{V})} \quad \forall \lambda.$$

With this setup, GLC makes sense, except...

Discrepancy: ${}^L\mathrm{Bun}$ is disconnected.

$$\boxed{\begin{aligned}\pi_0({}^L\mathrm{Bun}) &= H^2(C, \pi_1({}^LG)) = \pi_1({}^LG) = Z(G)^\wedge \Rightarrow \\ D^b({}^L\mathrm{Bun}, \mathcal{D}) &= \coprod_{\gamma \in Z(G)^\wedge} D^b({}^L\mathrm{Bun}^\gamma, \mathcal{D})\end{aligned}}$$

while Loc is irreducible $\Rightarrow D^b(\mathrm{Loc})$ indecomposable.

So: replace Loc by

$\boxed{\mathcal{L}oc := \text{moduli stack of semistable } G\text{-local systems}}$

$\mathcal{L}oc$ is an alg. stack w/ coarse moduli space = Loc.

$\exists$ open substack $\mathcal{L}oc^{rs}$ of "regularly stable" local systems

$\mathcal{L}oc^{rs} \to \mathrm{Loc}^{rs}$ : banded $Z(G)$-gerbe

$$\boxed{D^b(\mathcal{L}oc^{rs}) = \coprod_{\gamma \in Z(G)^\wedge} D^b(\mathcal{L}oc^{rs}, \gamma).}$$

Example: $G = {}^LG = GL(n)$

$\mathbb{V} = (V, \nabla)$ = rank $n$ loc. sys. / $C$

Hecke algebra $A$ is generated by $H^i$:

$\mathrm{Hecke}^i = \{(V, V', x) \mid V \subset V' \subset V(x),\ i = \ell(V'/V)\}$

fiber $= Gr(i, n)$.

Example: $G = GL(1)$, Bun = Pic

$$H^1 : C \times \mathrm{Pic}^d(C) \longrightarrow \mathrm{Pic}^{d+1}(C)$$
$$x, L \longmapsto L(x) \quad \text{abel-Jacobi}$$

$\mathbb{V} = (L, \nabla)$

$c(\mathbb{V})$ = the unique loc. sys. on $\mathrm{Pic}(C)$ restricting to $(L, \nabla)$.

$(\pi_1(\mathrm{Pic}^0 C) = \pi_1(C) / [\,,\,]\,.)$

$Higgs = Higgs_{C,G} = \{(V,\varphi) \mid \varphi : V \to V \otimes \omega_C\}$

$\boxed{\text{Hitchin map:}}$

$$\boxed{\begin{array}{l} h : Higgs_{C,G} \to B_{C,G} := H^0(C, (\omega_C \otimes \underline{t})/W) = \bigoplus_{i=1}^{r} H^0(C, \omega_C^{\otimes d_i}) \\ (V,\varphi) \mapsto \quad \text{"}\varphi \bmod W\text{"} \leftrightarrow (I_1(\varphi), \dots, I_r(\varphi)) \end{array}}$$

<u>Cameral covers:</u>

$$\boxed{\begin{array}{ccccc} C_b & \subset & \widetilde{C} & \to & Tot(C, \omega_C \otimes \underline{t}) \\ \downarrow & & \downarrow & & \downarrow \\ C \times \{b\} & \subset & C \times B & \to & Tot(C, (\omega_C \otimes \underline{t})/W) \end{array}}$$

$h^{-1}(b) \sim Prym(C_b / C)$

$h : Higgs \to B$ integrable system

$Higgs \supset T^* Bun$

$\forall \mu \Rightarrow$ associated <u>spectral covers</u> $\overline{C}_b^{\mu}$, $\overline{C}^{\mu} \to C \times B$.

e.g. $G = GL(n) \Rightarrow \deg(\overline{C}/C) = n$.

<u>[D & Gaitsgory '00]</u>: $\pi : \widetilde{C} \to C \times B$ determines an abelian group scheme $T$ over $C \times B$. $h : Higgs \to B$ is a principal homogeneous stack over the ab. group stack $Tors_T$ of $T$-torsors.

= "<u>abelianization</u>". (Hitchin, Atiyah, Beilinson-Kazhdan, Faltings, ...)

$\boxed{\text{Ancient approach}}$

$(G = GL(n))$

$$\boxed{\begin{array}{ccc} \overline{\mathcal{C}} & \longleftarrow & \overline{\mathcal{C}}_V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_V} \\ C\times B & \longleftarrow & C\times T^*_V \mathrm{Bun} \end{array}}$$

$\mathbb{V} = (V, \nabla) \Rightarrow$

$L$ on $\overline{\mathcal{C}}_V$, $\pi_{V*} L \simeq V \otimes \omega_C^{-(n-1)/2}$

$\nabla$ induces holo. conn. on $L$

(Depends on $\sqrt{\omega_C}$: use $\mathrm{Ram}_\varphi \simeq \pi_\varphi^* \omega_C^{\otimes (n-1)}$
The natural induced connection $\nabla'$ has residue $= \frac{1}{2}$
along $\mathrm{Ram}_\varphi$. Set $\nabla = \nabla' + \frac{1}{2} d\log f_\varphi$, where
$(f_\varphi) = -\mathrm{Ram}_\varphi + 2(n-1)\,\pi_\varphi^* \xi$, $\xi$ a section of $\sqrt{\omega_C}$)

Use case $G = GL(1)$: spread $(L, \nabla)$ to $(\tilde{L}, \tilde{\nabla})$ on
$\mathrm{Pic}\left(\overline{\mathcal{C}}_V / C\times T^*_V \mathrm{Bun}\right) =: \widetilde{\mathrm{Higgs}} = \mathrm{Higgs} \times_B T^*_V \mathrm{Bun}$

Now push to Higgs $\Rightarrow$

$(\tilde{V}, \tilde{\nabla})$ on Higgs, mero loc. sys along
fibers over $B$.

Fiber of $\tilde{V}$ at $(E, \psi) \in$ Higgs: $(E,\psi) \leftrightarrow (\overline{\mathcal{C}}_\psi, M_\psi)$, $\overline{\mathcal{C}}_\psi = \overline{\mathcal{C}}_\varphi$

$$\boxed{\tilde{V}\big|_{E,\psi} = \bigoplus_{\substack{\varphi \in T^*_V \mathrm{Bun} \\ h(\varphi) = h(\psi)}} \langle L_\varphi, M_\psi \rangle_{\overline{\mathcal{C}}_b}}$$

fiber of Poincaré bundle

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$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun} & & \mathrm{Bun} \times C \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Ab}\,\widetilde{\mathrm{Hecke}} \arrow[dl] \arrow[dr] & \\ \mathrm{Higgs} & & \mathrm{Higgs} \times C \end{tikzcd}$$

$(V, V', x, \beta) \ldots$

$$\left( (V,\varphi), (V',\varphi'), x, \begin{tikzcd} V|_{C-x} \arrow[r, "\beta"] \arrow[d, "\varphi"'] & V'|_{C-x} \arrow[d] \\ V \otimes \omega_C \arrow[r, "\beta\otimes \mathrm{id}"] & V' \otimes \omega_C \end{tikzcd} \right)$$

$Gr(i,n)$

finite collection of $\varphi_x$-eigenspaces

$\tilde{V}$ has an automorphic property w.r.t. $\widetilde{\mathrm{Hecke}}$.

To state this for all $G$, need: a duality between $\mathrm{Higgs}_G$, $\mathrm{Higgs}_{{}^LG}$.

To finish:

* Average over all $\varphi \in T^*_V \mathrm{Bun}$ ?
* Or: deform
* Or: use Simpson's.

# Non abelian Hodge theory

(Simpson, Corlette, ... )

$$\boxed{\mathrm{Higgs}^{\circ}(\mathcal{M}) \longleftrightarrow \mathrm{Loc}(\mathcal{M})}$$

$\mathcal{M}$ compact, Kahler.

Our case:

$\mathcal{M} = \mathrm{Bun}$

$\widetilde{\mathcal{M}} = \mathrm{Higgs} \sim T^*\mathrm{Bun}$

$H \subset \mathcal{M} \times \mathcal{M}$ Hecke

$\widetilde{H} := N^*_{H/\mathcal{M}\times\mathcal{M}} \subset \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$

Equivalences:

| sheaves on $\widetilde{\mathcal{M}}$ finite/$\mathcal{M}$ | $\underset{\text{spectral construction}}{\Longleftrightarrow}$ | Higgs sheaves on $\mathcal{M}$ | $\underset{\text{n.a.H.}}{\Longleftrightarrow}$ | Loc systems on $\mathcal{M}$ |
|---|---|---|---|---|
| $\uparrow$ $\widetilde{H}$ acts | | $\uparrow$ $H$ acts | | $\uparrow$ $H$ acts |

In general: actions are compatible

Our case

$$\boxed{\text{Key}: \ \widetilde{H} = \text{abelianized } \widetilde{\text{Hecke}}.}$$

Start: $\tilde{V}$ is an eigensheaf of $\tilde{H}$ on $\tilde{m}$

$\Leftrightarrow$

$\exists$ eigen Higgs sheaf on $m$

$\overset{?}{\Leftrightarrow}$

eigen local system on $m$ ?

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problem: $V$ defined only on Z-open in $m$
blows up along divisors:

* branch divisor
* non-very-stable: "wobbly".

Status:

Simpson: OK on curves
Boalch-Biquard
...
Mochizuki: OK w/ NCD, conditions.
(Hertling, Sabbah, ...)
Jost-Yang-Zuo

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$\boxed{\text{Key issue: geometry of wobblies.}}$

$E \to C$ V.B. is <u>very stable</u> if:

$$\boxed{\begin{array}{l}\varphi: E \to E \otimes K_C \\ \varphi \text{ nilpotent}\end{array}} \Rightarrow \boxed{\varphi = 0}$$

very stable $\Rightarrow$ stable

<u>nilpotent cone</u> $\boxed{\begin{array}{c} \mathcal{N} \subset T^*\mathrm{Bun} \\ \downarrow \\ \mathrm{Bun} \end{array}}$

$\mathcal{N} = \bigcup_i N^* S_i \qquad S_i \subset \mathrm{Bun} \qquad (S_0 = \mathrm{Bun})$

Non-very-stable $= \bigcup_{i \neq 0} S_i$

$\wr\wr$

$\boxed{\text{wobbly}}$

$\Rightarrow$ locus where automorphic Higgs sheaf on Bun blows up

$E \to X$ rank $n$ v.b.

$\alpha \in H^0(X, E)$ $\quad Y := (\alpha = 0) \subset X$ smooth

$$\Rightarrow \boxed{\begin{array}{l} \text{Koszul } (\Lambda^\bullet E, \wedge\alpha) \\ \quad \text{or}: (\Lambda^\bullet E^\vee, i_\alpha) \end{array}}$$

exact when $\alpha \neq 0$ ... $\Rightarrow$

$$\boxed{(\Lambda^\bullet E, \wedge\alpha) \simeq i_{Y*} \Lambda^\bullet F}$$

$$0 \to N^\vee \longrightarrow E|_Y \longrightarrow F \to 0 \qquad i_Y : Y \hookrightarrow X$$

case $E = \Omega^1_X$:

$$\boxed{\begin{array}{l} \mathbb{H}^k(X, (\Lambda^\bullet T_X, i_\alpha)) \simeq \bigoplus_{a+b=k} H^a(Y, \Lambda^b T_Y \otimes \det N) \\ \mathbb{H}^k(X, (\Omega^\bullet_X, \wedge\alpha)) \simeq \bigoplus_{a+b=k} H^a(Y, (\Omega^b_Y, 0)) \end{array}}$$

more generally:

$$\boxed{\begin{array}{lll} (V, \varphi) \text{ Higgs bundle} / X & & \\ \Leftrightarrow (C, L), & C \subset T^*X & \text{spectral cover} \\ & L \in \operatorname{Pic}(C) & \text{spectral sheaf} \end{array}}$$

$X_0 := C \cap_{T^*X} X$

$$\Rightarrow \boxed{\begin{array}{l} H^*_{\text{Higgs}}(X, (V, \varphi)) = \mathbb{H}^*(X, V \xrightarrow{\varphi} V \otimes \Omega^1 \xrightarrow{\varphi} V \otimes \Omega^2 \to \cdots) \\ \qquad = H^0(X_0, L \otimes \Lambda^\bullet V) \end{array}}$$

Apply this to $H, \widetilde{H}$ on $\mathcal{M}$, where

$X$ = fiber of $H$ ~ Grassmannian

$X_0$ = fiber of $\widetilde{H}$ ~ finite Grassmannian

(~ fixed points of $\varphi$)

$\Rightarrow$ $\boxed{H \text{ action on Higgs on } \mathcal{M}}$ is

compatible with

$\boxed{\widetilde{H} \text{ action on sheaves on } \widetilde{\mathcal{M}}}$.

Recent physics input:..

Kapustin - Witten
Gukov - Witten
Frenkel - Witten

interprets GLC in QFT

$\boxed{\text{A-branes} \Leftrightarrow \text{B-branes}}$

"mirror symmetry" | Wilson ops
't Hooft ops
....

arrives at similar conclusion:

$\boxed{\text{Abelianized GLC} = \text{classical limit of GLC}}$

but also

$\boxed{\Leftrightarrow \text{Full GLC}}$

via non-abelian Hodge on open manifolds.

## classical limit

**λ-connections:** $D(fs) = f Ds + \lambda\, df$

$D : V \to V \otimes \Omega^1$

$\lambda = 1$ : connection

$\lambda \neq 0$ : ditto

$\lambda = 0$ : Higgs field.

**Simpson:** diffeomorphism.

As $\lambda \to 0$: $(X = \mathrm{Bun}_{C, {}^LG})$

$$\boxed{\begin{array}{l} \mathrm{Loc}_X \rightsquigarrow \mathrm{Higgs}_X \\ \mathcal{D}_X\text{-modules} \rightsquigarrow \mathrm{Sym}^{\bullet} T_X\text{-modules} = \text{coherent sheaves on } T^*_X. \end{array}}$$

**classical limit of GLC:**

$$\boxed{D^b_{coh}(\mathcal{H}iggs^{\circ}_{C,G}) \cong D^b_{coh}(\mathrm{Higgs}_{C, {}^LG})}$$

note: need to understand $\lim_{\lambda \to 0} IC^{\lambda}$.

cf. Arinkin.

Classical limit conjecture:

$\exists$ natural equivalence of categories

$$\boxed{c: D^b(\mathcal{H}iggs) \xrightarrow{\sim} D^b({}^L\mathcal{H}iggs)}$$

inducing

$$\boxed{c^\circ: D^b(\mathcal{H}iggs^\circ) \xrightarrow{\sim} D^b({}^L\mathcal{H}iggs)}$$

$c^\circ$ sends str. sheaves of points in $\mathcal{H}iggs^\circ$ to Hecke eigensheaves:

$(V, \varphi) \in \mathcal{H}iggs^\circ \Rightarrow$

$$\boxed{H^\mu(c^\circ(V,\varphi)) = c^\circ(V,\varphi) \boxtimes (\rho^\mu(V), \rho^\mu(\varphi))}$$

D & Pantev 06/04/617: true over $B \setminus \Delta$.

$\Delta$ = discriminant, parametrizes singular cameral covers.

Underlying geometry: $\mathcal{H}iggs$, ${}^L\mathcal{H}iggs$ are dual integrable systems.

Hausel + Thaddeus: cases $GL(n)$, $SL(n)$.

Hitchin: $G_2$

Arinkin, Ngo: some info / $\Delta$

Ngo: $\Rightarrow$ "Fundamental Lemma"

## steps

$\boxed{\text{Duality along the base:}}$

$\exists$ isomorphism $\ell : B \xrightarrow{\sim} {}^L B$

$\ell(b) = {}^L b$

lifts to: $\ell : \widetilde{C} \xrightarrow{\sim} {}^L\widetilde{C}$

interchanges short + long roots.

$A, D, E, F, G$ self dual algebras $\Rightarrow$

$\ell : B \xrightarrow{\sim} B$.

$A, D, E$: $\ell = id$

$F, G$: $\ell \neq id$.

$$
\begin{tikzcd}
\widetilde{C} \arrow[rr] \arrow[d] & & \omega_C \otimes \underline{t} \arrow[r, "\sim"] \arrow[d] & \omega_C \otimes {}^L\underline{t} \arrow[d] & {}^L\widetilde{C} \arrow[l] \arrow[d] \\
B \times C = H^0(C, (\omega_C \otimes \underline{t})/W) \times C \arrow[rr, hook, "emb"] & & (\omega_C \otimes \underline{t})/W \arrow[r, "\sim"', "\kappa"] & (\omega_C \otimes {}^L\underline{t})/W & \cdots B \times C \arrow[l, hook']
\end{tikzcd}
$$

$\kappa$: any killing form $\underline{t} \xrightarrow{\sim} {}^L\underline{t}$

## Duality along fibers

$T_G = \Lambda \otimes \mathbb{C}^*$
$\Lambda = \text{cochar}_G$

[D+ Gaitsgory]:

$p : \tilde{C} \to C$ cameral $\Rightarrow$

$$\bar{T} := p_*(\Lambda \otimes \mathcal{O}^*_{\tilde{C}})^W$$

$$T := \{ t \in \bar{T} \mid \alpha(t)|_{D^\alpha} = 1 \ \forall \text{ root } \alpha \text{ of } \mathfrak{g} \}$$

$\alpha : T \to \mathbb{C}^* \qquad \alpha^\vee : \mathbb{C}^* \to T$

$D^\alpha \subset \tilde{C}$ : fixed divisor for reflection $s^\alpha$

$T^0 :=$ connected component of $T$

$T^0 \subset T \subset \bar{T}$ sheaves of ab. gps. on $C$

[DG]: $h^{-1}(b)$ is a torsor over $H^1(T)$

Real versions: $\mathcal{O}^*_{\tilde{C}} \leftrightarrow S^1$

$$\bar{T}_{\mathbb{R},s} = \{ \lambda \otimes z \mid \alpha^\vee(z^{\langle \alpha, \lambda \rangle}) = 1 \text{ in } T_{\mathbb{R}} \}$$

$\cup$

$$T_{\mathbb{R},s} = \{ \lambda \otimes z \mid z^{\langle \alpha, \lambda \rangle} = 1 \text{ in } S^1 \}$$

$\cup$

$$T^0_{\mathbb{R},s} = \{ \lambda \otimes z \mid \langle \alpha, \lambda \rangle = 0 \text{ in } \mathbb{Z} \}$$

Hodge theory from [DDP 2005]:

holo + real sheaves have same $H^1$!

reason: Hodge theory $\Rightarrow$ finite ker, coker but the cone is a complex of $\mathbb{R}$ v.s.'s.

$\Rightarrow$ topological calculations.

$$\boxed{\begin{array}{ccccc} \mathrm{Compo}^0 & & \mathrm{Compo} & & \overline{\mathrm{Compo}} \\ \uparrow & & \uparrow & & \uparrow \\ H^1(C, \mathcal{T}^0) & \to & H^1(C, \mathcal{T}) & \to & H^1(C, \overline{\mathcal{T}}) \\ \cup & & \cup & & \cup \\ P^0 & \longrightarrow & P & \longrightarrow & \overline{P} \quad \text{ab. var's} \end{array}}$$

$$\mathrm{Compo}^0 = \begin{cases} \mathbb{Z}/2 & G = Sp(N) \\ \pi_1(G) & \text{else} \end{cases}$$

$\mathrm{Compo} = \pi_1(G)$ always

so:

compo's of Hitchin fiber $\leftrightarrow$ compo's of Higgs.

**Key**: $$\boxed{\begin{aligned} \mathrm{Compo}^0 &= H^1(W, \Lambda^\vee)^\wedge \text{ for} \\ &= \left( \frac{(\Lambda^\vee)^W}{(1-\rho_1, \dots, 1-\rho_\ell)\Lambda^\vee} \right)^\wedge \text{ for} \end{aligned}}$$

$\Rightarrow$ $\mathrm{coker}(\rho) = \mathrm{coker}({}^t\rho)^\vee$; $\hat{\rho} = {}^t\rho$.

Duality for Higgs gerbes

Higgs is a banded $T$-gerbe
= over B, Higgs is a torsor over
the comm. gp. stack $Tors_T$.

$\exists$ Hitchin section $B \to$ Higgs $\Rightarrow$
it's the trivial torsor.