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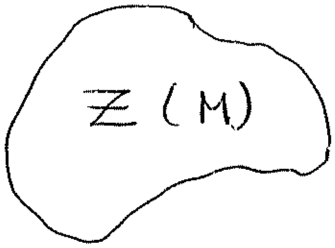
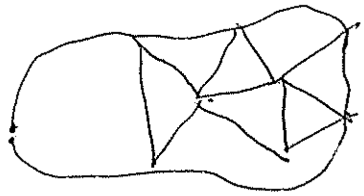
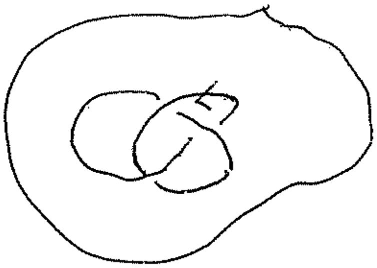
$sl(2;\mathbb{C})$ Chern-Simons theory ①







CS theory $\qquad \mathcal{U}_q(\mathfrak{g})$ quantum group

1) CS - theory
2) Geometric Quantization
3) Hyperbolic Geometry / Hikami's Invariants
4) Correspondence

$$I[A] = \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

A conn. on principal G-bdl

$$A \mapsto g^{-1} A g + g^{-1} dg$$

$su(2)$ $\qquad [T^a, T^b] = \varepsilon^{abc} T_c$

$$A = \sum_a A^a T^a$$

want to compute

$$\int DA \; e^{-i I[A]}$$

for $sl(2;\mathbb{C})$: A^a are complex 1-forms
2 copies : holomorphic + anti holomorphic part
(set coupling const for anti-holom. $= 0$)

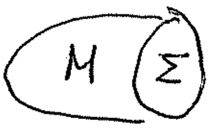
classical crit pts : flat connections
$$F = dA + A \wedge A$$

relation to gravity :
$$A = \omega + i e$$

<u>quantum theory</u> $\quad \hbar = \frac{\pi}{K}$

$$\underbrace{Z_\hbar(M) = \sum_\alpha e^{\frac{1}{\hbar} S_0}}_{\text{saddle pt approx}} \; \underbrace{- \frac{\delta}{2} \log\hbar + S_1 + \hbar S_2}_{\text{semi-class. contrib}}$$

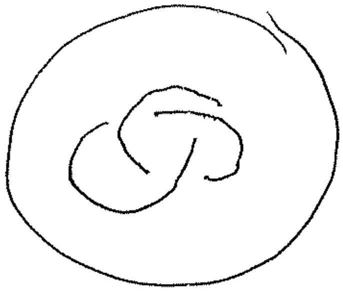
want to compute Z on knot-complem. in $S3$



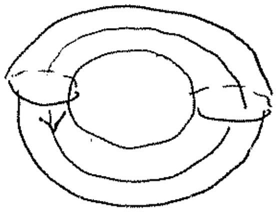
$$Z(M)(\text{bdy}) \in \mathcal{H}_\Sigma$$

$$= \langle Z(N) | Z(P) \rangle_{\mathcal{H}_\Sigma}$$

S^3



Z on solid T^3



2) <u>Geometric Quantization</u>

* Hilbert space for torus
  $\rightsquigarrow$ representation of π_1
  holonomies $\begin{pmatrix} l & * \\ 0 & 1/l \end{pmatrix}$ $\begin{pmatrix} m & * \\ 0 & 1/m \end{pmatrix}$
  $l = e^v \qquad m = e^u$

$\rightsquigarrow$ Poisson bracket $\qquad \{u, v\} = 1$

classical phase space $= \{(u,v)\} = \mathbb{C}^* \times \mathbb{C}^*$

$\rightsquigarrow$ quantization

$\hat{u}, \hat{v} \qquad [\hat{u}, \hat{v}] = -i\hbar$

$\mathcal{H}$ Hilbertspace

$$\hat{u}\,\Psi(u) = u\,\Psi(u)$$
$$\hat{v}\,\Psi(u) = i\hbar \frac{d}{du}\Psi(u)$$

$$\mathcal{H}_{T_2} = L^2(\mathbb{C}^*)$$

what are states ?

$(u_0, v_0) \in P$

$f(u,v) = 0 \quad \rightsquigarrow$ Lagr. submfld

$\rightsquigarrow \hat{f}(\hat{u},\hat{v})\,\Psi(u) = 0$

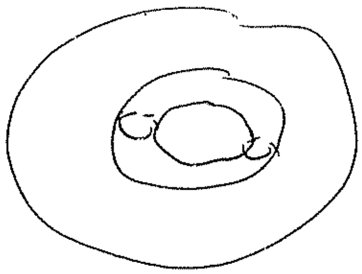
$p, q \qquad (\hat{p} - p_0)\,\psi(q) = 0 \Rightarrow \psi(q) = e^{i\frac{p_0}{\hbar}q}$

$\qquad\qquad (\hat{q} - q_0)\,\psi(q) = 0 \Rightarrow \psi = \delta(q - q_0)$

$\rightsquigarrow$ in our setting relation given by A-polynomial

$$A(l, m) = 0$$

unknot



$A(\text{unknot}) = l - 1$

$$A(3,1) = A(\text{trefoil}) = (l-1)(1+m^6 l)$$

$$A(4,1) = (l-1)(1-(1-m^2-2m^4-m^6+m^8)\,l+l^2)$$

$$\hat{A}(\hat{l},\hat{m})\, Z(M)(u) = 0$$

$$Z(M,u) = \sum_\alpha e^{\frac{1}{\hbar} S_0(u)} + \cdots$$

$$\hat{l} = e^{i\hbar \frac{d}{du}}$$

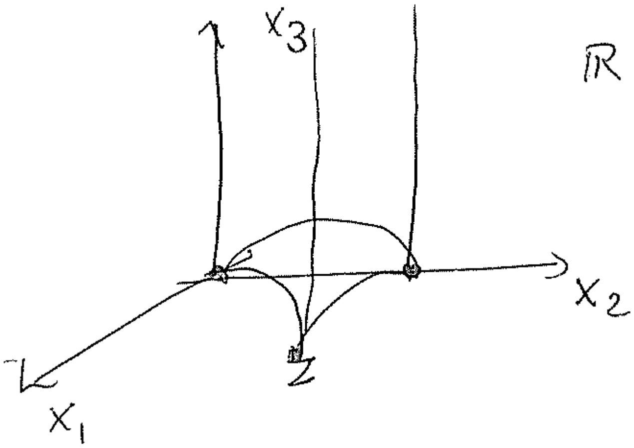
only choice ⇝ equation for S_0

$$A(e^{iS_0(u)}, e^u) = 0$$

once you fix branch,
perturbative expansion is determined

$$S_0^{\text{geom}} \sim \frac{1}{2} \operatorname{vol}(M,u)$$

3) Hyperbol. Geometry



$\mathbb{R}^3$

$$ds^2 = \frac{1}{x_3^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right)$$

- bdy S^2
- isometries $PSL(2,\mathbb{C})$

hyp. 3-mflds can be obtained as Knot/link complements in 3-mfld

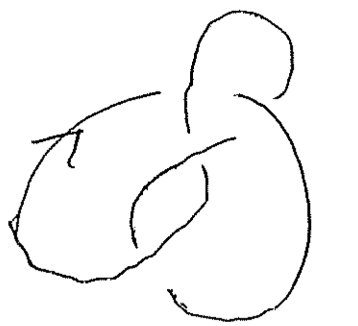
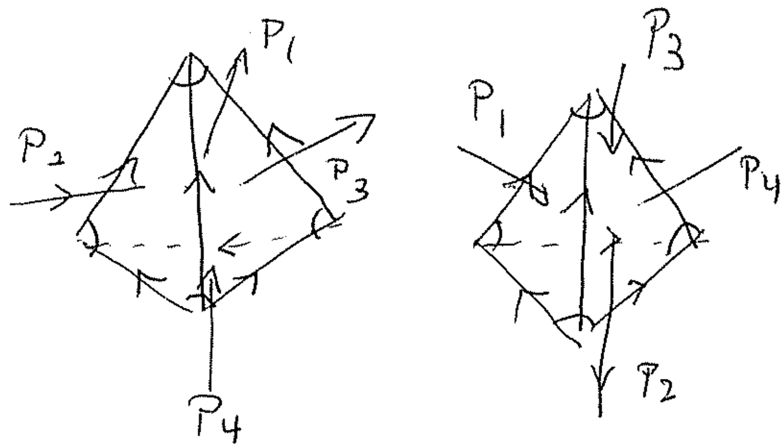
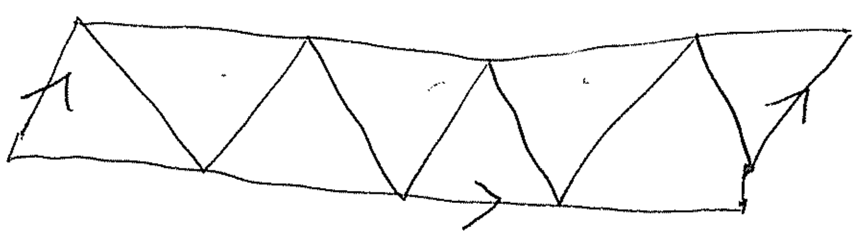


figure 8-Knot





$$\left[\begin{array}{r} f(z,w) = 0 \\ \frac{z}{w} = m \stackrel{c}{=} e^{2u} \end{array}\right]$$

for each face of tetra $\rightsquigarrow V$

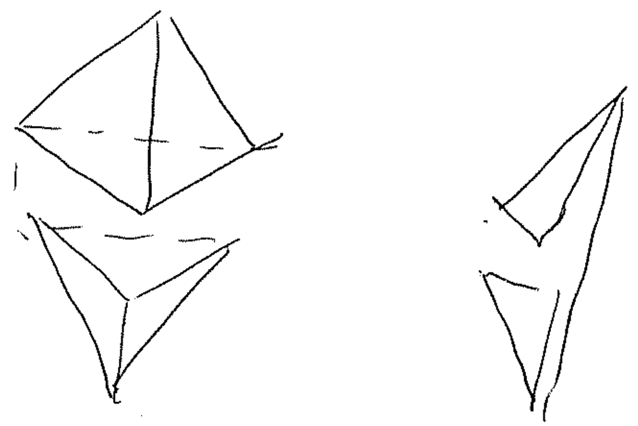
 $\sim V \otimes V \otimes V^* \otimes V^*$

S_{ij} : $V_i \otimes V_j - V_i \otimes V_j$

glueing by inner product

pentagon relation:

$$S_{23} S_{12} = S_{12} S_{13} S_{23}$$



$f(p_1, p_2)$

$$S = e^{\frac{1}{2i\hbar} \hat{q}_1 \hat{p}_2} \Phi_\hbar(\hat{p}_1 + \hat{q}_2 - \hat{p}_2)$$

$$\hat{q}_i \sim 2i\hbar \frac{d}{dp_i}$$

quantum dilogarithm

$$\Phi_\hbar(z) = e^{\frac{1}{4} \int_{\mathbb{R}+i\epsilon} \frac{dx}{x} \frac{e^{-ixz}}{\sinh(\pi v) \sinh(\hbar v)}}$$

$$\langle p_1 p_3 | S | p_2 p_4 \rangle \quad <'$$

$$\rightsquigarrow \frac{\delta(p_1 + p_3 - p_2)}{\sqrt{4\pi\hbar}} \quad \Phi_\hbar\left(p_4 - p_3 + i(\pi + \hbar)\right)$$

$$\langle p_4 p_2 | S^{-1} | p_3 \; p_1 \rangle$$

$$\rightsquigarrow \quad \delta(\ldots) \quad \frac{1}{\Phi_\hbar(\ldots)}$$

$$z = e^{p_4 - p_3} \qquad\qquad w = e^{p_1 - p_2}$$

$$Z(M) = \int dp \quad \prod \langle \ldots \rangle \quad \delta(\ldots)$$

$$\times \; \delta\left(p_4 - p_3 - p_1 + p_2 = -2u\right) \left(\frac{z}{w}\right)$$

$$= \frac{1}{4\pi\hbar} \int dp \quad \frac{\Phi_\hbar(p + i\pi + i\hbar)}{\Phi_\hbar(-p - 2u - i\pi - \hbar)}$$

$$e^{\frac{2i}{\hbar} u(u+p)}$$

$$\sim \quad e^{\frac{1}{2\hbar} \mathrm{Vol}(M, u)} \qquad \text{since} \quad \Phi_\hbar \sim e^{\frac{1}{2\hbar} \mathrm{Vol}(\Delta_z)}$$

define

$$Z(M) = \sum_{\substack{\text{saddle}\\ \beta}} e^{\frac{1}{\hbar} S_0^{\beta} - \frac{S^{\beta}}{2} \log \hbar + S_1(u)}$$

saddle pts match saddle pts in Chern-Simons-theory

$$Z_{\hbar}^{CS\ \alpha}(u) = \frac{c}{\hbar} e^{-u} Z_{\hbar}^{Hik\ \alpha}(u)$$

figure 8 : 2π

did computation for (5,2) knot as well